

## 5.1 - Radical Operations

Key

## Example 1 - Simplify

a) $4^2$	b) $16^{\frac{1}{2}}$	c) $\sqrt{16}$	d) $2^3$	e) $8^{\frac{1}{3}}$	f) $\sqrt[3]{8}$
$= 4(4)$	$= \sqrt{16}$	$= 4$	$= (2)(2)(2)$	$= \sqrt[3]{8}$	$= 2$
$= 16$	$= 4$		$= 8$	$= 2$	

What is the relationship between a, b, c? What is the relationship between d, e, f?  
 'square rooting' is the opposite of 'squaring'. Exponent of  $\frac{1}{2}$  same as 'square rooting'.  
 'cube rooting' is the opposite of 'cubing'. Exponent of  $\frac{1}{3}$  same as 'cube rooting'.

## RADICAL EXPRESSION

If  $x^n = a$ , then  $x = a^{\frac{1}{n}} = \sqrt[n]{a}$  ex. if  $x=2$  and  $n=3$   
 then  $2^3=8$  and  $8^{\frac{1}{3}}=2$

If 'a' and x are real numbers and n is a positive integer, then x is an  $n^{th}$  root of 'a' (ie  $x = a^{\frac{1}{n}} = \sqrt[n]{a}$ ) if  $x^n = a$

 $n^{th}$  root theorems:

1) If a is positive and n is even, then there exist TWO real  $n^{th}$  roots.

## Example 2 - Solve for x

a) $x^2 = 16$	b) $x^2 = 11$	c) $x^4 = 81$	d) $x^4 = 5$
$\sqrt{x^2} = \pm\sqrt{16}$	$\sqrt{x^2} = \pm\sqrt{11}$	$\sqrt[4]{x^4} = \pm\sqrt[4]{81}$	$\sqrt[4]{x^4} = \pm\sqrt[4]{5}$
$x = \sqrt{16} = 4$ and $x = -\sqrt{16} = -4$	$x = \sqrt{11}$ and $x = -\sqrt{11}$	$x = \sqrt[4]{81} = 3$ $x = -\sqrt[4]{81} = -3$	$x = \sqrt[4]{5}$ $x = -\sqrt[4]{5}$
b/c $4^2 = 16$ and $(-4)^2 = 16$	b/c $(-\sqrt{11})^2 = (\sqrt{11})^2 = 11$	b/c $(-3)^4 = (3)^4 = 81$	$(-\sqrt[4]{5})^4 = (\sqrt[4]{5})^4 = 5$

2) If a is negative and n is even, then there are NO real number solutions.

## Example 3 - Solve for x

a)  $x^2 = -25$

$x = \pm\sqrt{-25}$

$x = \emptyset$

b)  $x^4 = -7$

$x = \pm\sqrt[4]{-7}$

$x = \emptyset$

3) If  $n$  is odd, then there is ONE real  $n^{th}$  root of  $a$ .

Example 4 – Solve for  $x$

a)  $x^3 = 8$

$$\sqrt[3]{x^3} = \sqrt[3]{8}$$

$$x = 2$$

b/c

$$(2)(2)(2) = 8$$

b)  $x^3 = -8$

$$\sqrt[3]{x^3} = \sqrt[3]{-8}$$

$$x = -2$$

b/c

$$(-2)(-2)(-2) = -8$$

c)  $x^5 = -4$

$$\sqrt[5]{x^5} = \sqrt[5]{-4}$$

$$x = \sqrt[5]{-4}$$

b/c

$$(\sqrt[5]{-4})^5 = -4$$

4) If  $a$  is zero, then there is ONE real  $n^{th}$  root of  $a$ , and it is ZERO.

Example 5 – Solve for  $x$

a)  $x^2 = 0$

$$x = \pm\sqrt{0}$$

$$x = 0$$

b)  $x^5 = 0$

$$x = \sqrt[5]{0}$$

$$x = 0$$

### Radical Properties from Math 10:

1)  $a^{\frac{1}{n}} = \sqrt[n]{a}$  as discussed in above notes

2)  $a^{\frac{m}{n}} = (a^{\frac{1}{n}})^m = (\sqrt[n]{a})^m$  Example:  $16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = (2)^3 = 8$

3)  $a^{-\frac{m}{n}} = (a^{\frac{1}{n}})^{-m} = (\sqrt[n]{a})^{-m} = \frac{1}{(\sqrt[n]{a})^m}$  Example:  $27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{3^2} = \frac{1}{9}$

4)  $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$  Example:  $\sqrt[3]{\frac{27}{64}} = \frac{\sqrt[3]{27}}{\sqrt[3]{64}} = \frac{3}{4}$

5)  $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$  Example:  $\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2 \cdot \sqrt{3} = 2\sqrt{3}$

**absolute value**

Absolute Value is "how many jumps the number is from zero". Stated another way, it is the distance from zero on the number line, regardless of direction. Distances are always POSITIVE values. 3 is 3 jumps from zero, so the absolute value of 3, or  $|3| = 3$ . -3 is 3 jumps from zero, so  $|-3| = 3$ . So if  $|x| = 3$ ,  $x$  could have been 3 OR -3. For every absolute value solution, there is a positive and negative possibility.

Example 1 - Evaluate

$$\begin{array}{lllll} \text{a) } |5| & \text{b) } |-7| & \text{c) } |-0.34| & \text{d) } \left| \frac{5}{6} \right| & \text{e) } \left| -6\frac{3}{8} \right| \\ = 5 & = 7 & = 0.34 & = \frac{5}{6} & = 6\frac{3}{8} \end{array}$$

Example 2 - What are the possible values of  $x$ ?

$$\begin{array}{lll} \text{a) } |x| = 6 & \text{b) } |x| = 9.7 & \text{c) } |x| = -2 \\ x = \pm 6 & x = \pm 9.7 & x = \emptyset \end{array}$$

**Radical Review**

Example 3 - Identify and define all parts of the radical, then simplify:

$$\begin{array}{c} \text{coefficient} \swarrow \quad \text{index} \searrow \\ \sqrt[3]{5 \cdot 8} \\ \text{radical sign} \quad \text{radicand} \end{array} = 5(2) = 10$$

Roots of Positive Powers of  $x$ :

Case 1: When  $x \geq 0$  in  $\sqrt[n]{x^n}$  with  $n$  a positive integer.

The square roots of negative numbers are undefined in the set of real numbers. Therefore, if  $x \geq 0$ , simplification is easier to realize.

For example:

$$\begin{array}{ll} \sqrt{x^2} = \sqrt{(x \cdot x)} = x & \sqrt[3]{x^2} = \sqrt[3]{x \cdot x} = \sqrt[3]{x^2} \\ \sqrt{x^3} = \sqrt{(x \cdot x) \cdot x} = x\sqrt{x}, & \sqrt[3]{x^3} = \sqrt[3]{(x \cdot x) \cdot x} = x \\ \sqrt{x^4} = \sqrt{(x \cdot x)(x \cdot x)} = x^2 & \sqrt[3]{x^4} = \sqrt[3]{(x \cdot x) \cdot x^2} = x\sqrt[3]{x^2} \\ \sqrt{x^5} = \sqrt{(x \cdot x)(x \cdot x)x} = x^2\sqrt{x}, & \sqrt[3]{x^5} = \sqrt[3]{(x \cdot x) \cdot x^2} = x\sqrt[3]{x^2} \\ \sqrt{x^6} = \sqrt{(x \cdot x)(x \cdot x)(x \cdot x)} = x^3 & \sqrt[3]{x^6} = \sqrt[3]{(x \cdot x) \cdot x^2} = x^2 \end{array}$$

Example 4 - Simplify. Assume all variables represent positive numbers

$$\begin{array}{lllll} \text{a) } \sqrt{16y^2} & \text{b) } \sqrt{x^4y^3} & \text{c) } \sqrt{25x^5y^3z^2} & \text{d) } \sqrt[3]{8x^4y^5} & \text{e) } \sqrt[3]{27x^3y^6} \\ = \sqrt{4 \cdot 4 \cdot y \cdot y} & = \sqrt{(x \cdot x)(x \cdot x)(y \cdot y)y} & = \sqrt{5 \cdot 5(x \cdot x)(x \cdot x)x(y \cdot y)y(z \cdot z)} & = 5x^2y^2\sqrt{xy} & = \sqrt[3]{3 \cdot 3 \cdot 3(x \cdot x \cdot x)(y \cdot y \cdot y)(y \cdot y \cdot y)} \\ = 4y & = x^2y\sqrt{y} & \downarrow & = \sqrt[3]{2 \cdot 2 \cdot 2(x \cdot x \cdot x)x(y \cdot y \cdot y)y \cdot y} \\ & & & & = 2xy\sqrt[3]{xy^2} \end{array}$$

## 5.2 - Simplifying Radicals

Radicals can be written as fractional exponents, as learned in Math 10.

Examples:  $\sqrt{2} = 2^{\frac{1}{2}}$        $\sqrt[3]{x} = x^{\frac{1}{3}}$       Generally:  $\sqrt[n]{a} = a^{\frac{1}{n}}$

Three Important Relationships of Radicals, all from Math 10:

1)  $\sqrt[n]{a^n} = a$ ,  $a \geq 0$  because  $\sqrt[n]{a^n} = (a^n)^{\frac{1}{n}} = a$  Example:  $\sqrt[2]{5^2} = 5^{\frac{2}{2}} = 5$

2)  $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$ ,  $a, b \geq 0$  because  $\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}} = \sqrt[n]{a} \times \sqrt[n]{b}$

3)  $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ ,  $a \geq 0, b > 0$  because  $\sqrt[n]{\frac{a}{b}} = (\frac{a}{b})^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

### Simplifying Expressions Containing Radicals

Example 1 – Simplify  $\sqrt{20}$

$$\begin{aligned} ① \quad & \sqrt{20} \\ &= \sqrt{4 \cdot 5} \\ &= \sqrt{4} \cdot \sqrt{5} \\ &= 2 \cdot \sqrt{5} \\ &= 2\sqrt{5} \end{aligned}$$

Two Methods:

$$\begin{aligned} & \frac{\sqrt{20}}{\sqrt{20}} \\ &= \frac{\sqrt{4 \cdot 5}}{\sqrt{(2 \cdot 2) \cdot 5}} \\ &= \frac{2\sqrt{5}}{2\sqrt{5}} \end{aligned}$$

It is beneficial to know the perfect squares up to 144, perfect cubes up to 125, and perfect fourths up to 81.

P.S. : 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144

P.C. : 1, 8, 27, 64, 125

P.F. : 1, 16, 81

Example 2 – Simplify

a) $\sqrt{8}$	b) $\sqrt{27}$	c) $3\sqrt{52}$	d) $\sqrt[3]{24}$	e) $\sqrt[5]{81}$	f) $\sqrt[4]{32}$
$= \sqrt{4 \cdot 2}$	$= \sqrt{9 \cdot 3}$	$= 3 \cdot \sqrt{4 \cdot 13}$	$= \sqrt[3]{8 \cdot 3}$	$= 5 \cdot \sqrt[3]{27 \cdot 3}$	$= \sqrt[4]{16 \cdot 2}$
$= \sqrt{4} \cdot \sqrt{2}$	$= \sqrt{9} \cdot \sqrt{3}$	$= 3 \cdot \sqrt{4} \cdot \sqrt{13}$	$= \sqrt[3]{8} \cdot \sqrt[3]{3}$	$= 5 \cdot \sqrt[3]{27} \cdot \sqrt[3]{3}$	$= \sqrt[4]{16} \cdot \sqrt[4]{2}$
$= 2\sqrt{2}$	$= 3\sqrt{3}$	$= 3 \cdot 2 \cdot \sqrt{13}$	$= 2 \cdot \sqrt[3]{3}$	$= 5 \cdot 3 \cdot \sqrt[3]{3}$	$= 2\sqrt[4]{2}$
		$= 6\sqrt{13}$	$= 2\sqrt[3]{3}$	$= 15\sqrt[3]{3}$	

Example 3 – Simplify (assume variables are positive)

$$\begin{aligned} \text{a) } & \sqrt{18x^3y^6} \\ & = \sqrt{9 \cdot 2 \cdot x^2 \cdot x \cdot y^6} \\ & = 3xy^3\sqrt{2x} \end{aligned}$$

$$\begin{aligned} \text{b) } & \sqrt{63n^7p^4} \\ & = \sqrt{9 \cdot 7n^6n p^4} \\ & = 3n^3p^2\sqrt{7n} \end{aligned}$$

$$\begin{aligned} \text{c) } & \sqrt{32x^8y^{11}} \\ & = \sqrt{16 \cdot 2x^8y^{10}y} \\ & = 4x^4y^5\sqrt{2y} \end{aligned}$$

$$\begin{aligned} \text{d) } & \sqrt[3]{40a^4b^8c^{15}} \\ & = \sqrt[3]{8 \cdot 5a^3a b^6b^2c^{15}} \\ & = 2ab^2c^5\sqrt[3]{5ab^2} \end{aligned}$$

$$\begin{aligned} \text{e) } & \sqrt[3]{54a^5b^{10}} \\ & = \sqrt[3]{27 \cdot 2a^3a^2b^9b} \\ & = 3ab^3\sqrt[3]{2a^2b} \end{aligned}$$

$$\begin{aligned} \text{f) } & \sqrt[4]{m^7} \\ & = \sqrt[4]{m^4 \cdot m^3} \\ & = m\sqrt[4]{m^3} \end{aligned}$$

$$\begin{aligned} \text{g) } & \sqrt[4]{162x^3y^{11}z^5} \\ & = \sqrt[4]{81 \cdot 2x^3y^8y^3z^4z} \\ & = 3y^2\sqrt[4]{22^3y^3z} \end{aligned}$$

$$\begin{aligned} \text{h) } & \sqrt[3]{\frac{x^{13}}{64}} \\ & = \sqrt[3]{\frac{x^{12}x}{64}} \\ & = \frac{x^4\sqrt[3]{x}}{4} \end{aligned}$$

### Changing Mixed Radicals to Entire Radicals

Example 4 – Change to Entire (assume variables are positive)

$$\begin{aligned} \text{a) } & 4\sqrt{3} \\ & = \sqrt{16} \cdot \sqrt{3} \\ & = \sqrt{16 \cdot 3} \\ & = \sqrt{48} \end{aligned}$$

$$\begin{aligned} \text{b) } & 3\sqrt{5} \\ & = \sqrt{9} \cdot \sqrt{5} \\ & = \sqrt{9 \cdot 5} \\ & = \sqrt{45} \end{aligned}$$

$$\begin{aligned} \text{c) } & 2\sqrt[3]{7} \\ & = \sqrt[3]{8} \cdot \sqrt[3]{7} \\ & = \sqrt[3]{8 \cdot 7} \\ & = \sqrt[3]{56} \end{aligned}$$

$$\begin{aligned} \text{d) } & -2x\sqrt{6x} \\ & = -1 \cdot 2x \cdot \sqrt{6x} \\ & = -1 \sqrt{4x^2} \cdot \sqrt{6x} \\ & = -\sqrt{24x^3} \end{aligned}$$

$$\begin{aligned} \text{e) } & x^3\sqrt{x} \\ & = \sqrt{x^6} \cdot \sqrt{x} \\ & = \sqrt{x^6 \cdot x} \\ & = \sqrt{x^7} \end{aligned}$$

$$\begin{aligned} \text{f) } & 3a^2b\sqrt[3]{b^2c} \\ & = 3a^2b \cdot \sqrt[3]{b^2c} \\ & = \sqrt[3]{27a^6b^3} \cdot \sqrt[3]{b^2c} \\ & = \sqrt[3]{27a^6b^5c} \end{aligned}$$

$$\begin{aligned} \text{g) } & \frac{3x^2y}{5}\sqrt[3]{2xy^2} \\ & = \frac{3x^2y}{5} \cdot \sqrt[3]{2xy^2} \\ & = \sqrt[3]{\frac{27x^6y^3}{125}} \cdot \sqrt[3]{2xy^2} \\ & = \sqrt[3]{\frac{54x^7y^5}{125}} \end{aligned}$$

## 5.3. - Adding and Subtracting Radical Expressions

### Like Radicals

'Like Radicals' work very similar to 'Like Terms'.

$$\text{Simplify: } 3x + 2x = 5x$$

$$\text{Simplify } 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$$

Like radicals have the same index and same radicand

Steps for adding & subtracting like radicals:

- ① Check if radicals are 'like'. If not, do any simplifying
- ② Add or subtract coefficients, leave radicals the same.

### Example 1 - Simplify

$$a) 7\sqrt{3} - 2\sqrt{3}$$

$$5\sqrt{3}$$

$$b) -5\sqrt[3]{10} - 6\sqrt[3]{10}$$

$$-11\sqrt[3]{10}$$

$$c) 4\sqrt{2} - 5\sqrt{2}$$

cannot  
simplify

$$d) 2\sqrt{75} + 3\sqrt{3}$$

$$2\sqrt{25 \cdot 3} + 3\sqrt{3}$$

$$10\sqrt{3} + 3\sqrt{3}$$

$$13\sqrt{3}$$

$$e) -\sqrt{27} + 3\sqrt{5} - \sqrt{80} - 2\sqrt{12}$$

$$-3\sqrt{3} + 3\sqrt{5} - 4\sqrt{5} - 4\sqrt{3}$$

$$-7\sqrt{3} - \sqrt{5}$$

$$f) \sqrt{9b} - 3\sqrt{16b}, b \geq 0$$

$$3\sqrt{b} - 12\sqrt{b}$$

$$= -9\sqrt{b}, b \geq 0$$

In example f, why does  $b$  have to be greater than or equal to zero? If it was negative, the radicands would become negative

**Example 2 – Simplify**

a)  $\sqrt{27xy} + \sqrt{8xy}$

$$3\sqrt{3xy} + 2\sqrt{2xy}$$

b)  $4\sqrt[3]{16} + 3\sqrt[3]{54}$

$$8\sqrt[3]{2} + 9\sqrt[3]{2}$$
$$= 17\sqrt[3]{2}$$

c)  $3x\sqrt{63y} - 5\sqrt{28x^2y}; \quad y \geq 0$

$$9x\sqrt{7y} - 10x\sqrt{7y}$$

$$-x\sqrt{7y}$$

d)  $\frac{5}{2}\sqrt[3]{16x^4y^5} - xy\sqrt[3]{54xy^2}$

$$\left(\frac{5}{2}\right)2xy\sqrt[3]{2xy^2} - 3xy\sqrt[3]{2xy^2}$$
$$= 5xy\sqrt[3]{2xy^2} - 3xy\sqrt[3]{2xy^2}$$
$$= 2xy\sqrt[3]{2xy^2}$$

## 5.4A - Multiplying & Dividing Radical Expressions

multiplying  
radicals

$$\begin{aligned} \text{Example 1 - Multiply } 2\sqrt{5} (3\sqrt{5}) &\quad \text{Verify your answer: } 2\sqrt{5} (3\sqrt{5}) \\ &= 2 \cdot \sqrt{5} \cdot 3 \cdot \sqrt{5} \\ &= 2 \cdot 3 \cdot \sqrt{5} \cdot \sqrt{5} \\ &= 6 \cdot \sqrt{5 \cdot 5} = 6\sqrt{25} = 6(5) = 30 &= 30 \checkmark \end{aligned}$$

To multiply radicals:

- ① Multiply coefficients
- ② Multiply radicands (if index is the same)
- ③ Simplify

In general:  $(x\sqrt[n]{a})(y\sqrt[n]{b})$

$$= xy\sqrt[n]{ab}$$

where  $n$  is a natural number and  $x, y, a, b$  are real numbers. If  $n$  is even,  $a \geq 0$  and  $b \geq 0$

Example 2 - Simplify: a)  $5\sqrt{3} (\sqrt{6})$

$$\begin{aligned} &= 5\sqrt{18} \\ &= 15\sqrt{2} \end{aligned}$$

b)  $2\sqrt{6} (4\sqrt{8})$

$$\begin{aligned} &= 8\sqrt{48} \\ &= 32\sqrt{3} \end{aligned}$$

c)  $-3\sqrt{2x} (4\sqrt{3x}) \quad x \geq 0$

$$-12\sqrt{6x^2}$$

$$-12x\sqrt{6}$$

$$\text{d) } -2\overbrace{\sqrt[3]{11}}^{\text{brace}} (4\overbrace{\sqrt[3]{2}}^{\text{brace}} - 3\overbrace{\sqrt[3]{3}}^{\text{brace}})$$

$$-8\sqrt[3]{22} + 6\sqrt[3]{33}$$

e)  $(4\sqrt{2} + 3)(\sqrt{7} - 5\sqrt{14}) \quad \text{FOIL}$

$$4\sqrt{14} - 20\sqrt{28} + 3\sqrt{7} - 15\sqrt{14}$$

$$4\sqrt{14} - 40\sqrt{7} + 3\sqrt{7} - 15\sqrt{14}$$

$$-11\sqrt{14} - 37\sqrt{7}$$

f)  $(\sqrt[3]{x} + \sqrt[3]{y})(\sqrt[3]{x^2} - \sqrt[3]{xy} + \sqrt[3]{y^2})$

$$\sqrt[3]{x^3} - \underbrace{\sqrt[3]{x^2}y}_{\text{brace}} + \underbrace{\sqrt[3]{xy^2}}_{\text{brace}} + \underbrace{\sqrt[3]{y^2}y}_{\text{brace}} - \underbrace{\sqrt[3]{xy^2}}_{\text{brace}} + \underbrace{\sqrt[3]{y^3}}_{\text{brace}}$$

$$= x + y$$

dividing  
radicals

Example 3 - Divide  $\frac{6\sqrt{12}}{3\sqrt{6}}$

$$= 2\sqrt{2}$$

Verify your answer:  $\frac{6\sqrt{12}}{3\sqrt{6}} = \frac{20.785}{7.348}$

$$= 2.828$$

To divide radicals:

① Divide coefficients

(same as  $2\sqrt{2}$ )

② Divide radicands (if index the same)

③ Simplify if possible

In general:

$$\frac{x\sqrt{a}}{y\sqrt{b}} = \frac{x}{y} \sqrt{\frac{a}{b}} \quad \text{with same stipulations as multiplying radicals.}$$

also:  $y \neq 0, b \neq 0$  ( $a, b > 0$ )

Example 4 - Simplify: a)  $\frac{-24\sqrt[3]{14}}{8\sqrt[3]{2}}$

$$-3\sqrt[3]{7}$$

b)  $\frac{2\sqrt{51}}{\sqrt{3}}$

$$2\sqrt{17}$$

c)  $\frac{\sqrt{18x^3}}{\sqrt{3x}}, x > 0$

$$\sqrt{6x^2}$$

$$= x\sqrt{6}$$

### Multiplying & Dividing Terms with Different Indices

Example 5 - Simplify  $\sqrt[2]{x^3} (\sqrt[3]{x}), x \geq 0$

$$= x^{\frac{3}{2}} \cdot x^{\frac{1}{3}} \quad \frac{3}{2} + \frac{1}{3} = \frac{11}{6}$$

$$= x^{\frac{11}{6}}$$

$$= \sqrt[6]{x^6 x^5}$$

$$= x \sqrt[6]{x^5}$$

Example 6 - Simplify  $\frac{\sqrt[2]{x^3}}{\sqrt[3]{x}}, x > 0$

$$x^{\frac{3}{2}} \div x^{\frac{1}{3}} \quad \frac{3}{2} - \frac{1}{3} = \frac{7}{6}$$

$$= x^{\frac{7}{6}}$$

$$= \sqrt[6]{x^6 x} = x \sqrt[6]{x}$$

## 5.4B - Rationalizing the Denominator

### rationalizing the denominator

A final answer cannot have a radical in the denominator. Therefore, you may have to 'rationalize the denominator' – a process that will eliminate the radical from the denominator without changing the value of the expression.

Example 1 - Rationalize: a)  $\frac{3 + \sqrt{5}}{\sqrt{5} + \sqrt{5}}$    b)  $\sqrt{\frac{2}{7}}$    c)  $\frac{6 + \sqrt{5}}{7\sqrt{5} + \sqrt{5}}$    d)  $\frac{2\sqrt{5} + (\sqrt[3]{6})^2}{3\sqrt[3]{6} + (\sqrt[3]{6})^2}$

If the denominator is a radical monomial, multiply the numerator and denominator by that radical.

$$\begin{aligned} &= \frac{3\sqrt{5}}{5} \quad = \frac{\sqrt{2} \cdot \sqrt{7}}{\sqrt{7} \cdot \sqrt{7}} \quad = \frac{6\sqrt{5}}{7(5)} \quad = \frac{2\sqrt{5}(\sqrt[3]{6})^2}{3(\sqrt[3]{6})^2} \\ &= \frac{\sqrt{14}}{7} \quad = \frac{6\sqrt{5}}{35} \quad = \frac{2\sqrt{5}(\sqrt[3]{6})^2}{18} \\ &= \frac{\sqrt{5}(\sqrt[3]{6})^2}{9} \end{aligned}$$

e)  $\sqrt[3]{\frac{2}{y}}$

$$= \frac{\sqrt[3]{2} \cdot (\sqrt[3]{y})^2}{\sqrt[3]{y} \cdot (\sqrt[3]{y})^2}$$

$$= \frac{\sqrt[3]{2y^2}}{y}$$

f)  $\frac{\sqrt[4]{5x} \cdot 4\sqrt{10x^3})^3}{\sqrt[4]{10x^3} \cdot (4\sqrt{10x^3})^3}$

$$= \frac{4\sqrt{5000x^{10}}}{10x^3}$$

$$= \frac{4\sqrt{625 \cdot 8 \cdot x^8 \cdot x^2}}{10x^3}$$

$$= \frac{5x^2\sqrt[4]{8x^2}}{10x^3}$$

$$= \frac{4\sqrt{8x^2}}{2x}$$

Example 2 – Rationalize: a)  $\frac{3}{\sqrt{x}-2}$

If the denominator is a radical binomial, multiply the numerator & denominator by its conjugate.

conjugate of

$\sqrt{x}-2$  is  $\sqrt{x}+2$

$$\frac{3}{\sqrt{x}-2} \cdot \frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}$$

$$\frac{3(\sqrt{x}+6)}{x-2\sqrt{x}+2\sqrt{x}-4}$$

$$= \frac{3\sqrt{x}+6}{x-4}$$

b)  $\frac{2+\sqrt{2}}{3\sqrt{5}-4} (3\sqrt{5}+4)$

$$\frac{6\sqrt{5} + 8 + 3\sqrt{10} + 4\sqrt{2}}{9(5) - 12\sqrt{5} + 12\sqrt{5} - 16}$$

$$\frac{6\sqrt{5} + 3\sqrt{10} + 4\sqrt{2} + 8}{45 - 16}$$

$$\frac{6\sqrt{5} + 3\sqrt{10} + 4\sqrt{2} + 8}{29}$$

$$\begin{aligned}
 c) & \frac{\sqrt{a} + \sqrt{2b}}{\sqrt{a} - \sqrt{2b}}, a, b \geq 0 \\
 & \frac{\sqrt{a} + \sqrt{2b}}{\sqrt{a} - \sqrt{2b}} \cdot \frac{(\sqrt{a} + \sqrt{2b})}{(\sqrt{a} + \sqrt{2b})} \\
 & = \frac{a + \sqrt{2ab} + \sqrt{2ab} + 2b}{a - \sqrt{2ab} + \sqrt{2ab} - 2b} \\
 & = \frac{a + 2\sqrt{2ab} + 2b}{a - 2b}, [a \neq 2b]
 \end{aligned}$$

Example 3 - Simplify

$$\begin{aligned}
 a) & 6\sqrt{\frac{3}{4x}}, x > 0 & b) & \frac{-7}{2\sqrt[3]{9p}}, p \neq 0 \\
 & = \frac{6\sqrt{3}}{\sqrt{4x}} & & = \frac{-7 \cdot (\sqrt[3]{9p})^2}{2\sqrt[3]{9p} \cdot (\sqrt[3]{9p})^2} \\
 & = \frac{6\sqrt{3}}{2\sqrt{x}} & & = \frac{-7(\sqrt[3]{9p})^2}{2(9p)} \\
 & = \frac{3\sqrt{3} \cdot \sqrt{x}}{\sqrt{x} \cdot \sqrt{x}} & & = \frac{-7(\sqrt[3]{9p})^2}{18p} \\
 & = \frac{3\sqrt{3x}}{x} & & = \frac{-7(\sqrt[3]{81p^2})}{18p} \\
 & & & = \frac{-7(3\sqrt[3]{3p^2})}{18p}
 \end{aligned}$$

Example 4 – The surface area of a sphere is  $S = 4\pi r^2$ . If the surface area of the sphere is  $144 \text{ mm}^2$ , what is the radius?

$$\begin{aligned}
 144 &= 4\pi r^2 & r &= \sqrt{\frac{36}{\pi}} \\
 r^2 &= \frac{144}{4\pi} & r &= \frac{6 \cdot \sqrt{\pi}}{\sqrt{\pi} \cdot \sqrt{\pi}} \\
 r^2 &= \frac{36}{\pi} & r &= \frac{6\sqrt{\pi}}{\pi} \\
 r &= \pm \sqrt{\frac{36}{\pi}}
 \end{aligned}$$

reject the negative  
radius

## 5.5 - Radical Equations

**Radical Equations** are mathematical equations that include a radical, such as:

$2\sqrt{6x} - 1 = 11$ . If the index of the radical is even, there are restrictions on the variable. Since it is not possible to find the square root of a negative number, the radicand cannot be negative.

Example 1 – Find the restriction on the variable:

a)  $2\sqrt{6x} - 1 = 11$

$$6x \geq 0$$

$$x \geq \frac{0}{6}; \quad \boxed{x \geq 0}$$

b)  $\sqrt{x+2} = 49$

$$x+2 \geq 0$$

$$\boxed{x \geq -2}$$

c)  $7\sqrt{-2x+3} = 35$

$$-2x+3 \geq 0$$

$$-2x \geq -3$$

$$\boxed{x \leq \frac{3}{2}}$$

d)  $\sqrt{3x+4} = \sqrt{2x-4}$

$$3x+4 \geq 0$$

$$2x-4 \geq 0$$

$$3x \geq -4$$

$$x \geq -\frac{4}{3}$$

$$2x \geq 4$$

$$x \geq 2$$

so  $\boxed{x \geq 2}$  as anything between  $-\frac{4}{3}$  and 2 will not work for the 2nd radical.

Steps to solving radical equations:

- Find the restrictions on the variable in the radicand (if the index is even). Remember, the radicand must be set to  $\geq 0$  and then solved (if you multiply or divide by a negative number to both sides, FLIP the inequality).
- Get the radical all by itself on one side of the equation.
- If the index is 2, square both sides (if index is 3, cube both sides, etc.) and then solve for the variable.
- See if the solution is affected by the restriction.
- Check the answer using the original equation to see if solutions are valid or extraneous.

Example 2 – Solve a)  $2\sqrt{6x} - 1 = 11$

$$2\sqrt{6x} - 1 = 11 \quad \boxed{x \geq 0}$$

$$\frac{2\sqrt{6x}}{2} = \frac{12}{2}$$

$$\sqrt{6x} = 6$$

$$(\sqrt{6x})^2 = 6^2$$

$$6x = 36$$

$$x = 6$$

check:

$$2\sqrt{6(6)} - 1 = 11$$

$$2\sqrt{36} - 1 = 11$$

$$2(6) - 1 = 11$$

$$12 - 1 = 11$$

$$\checkmark$$

b)  $-8 + \sqrt{\frac{3y}{5}} = 2$

$$\sqrt{\frac{3y}{5}} = 10 \quad \frac{3y}{5} \geq 0$$

$$(\sqrt{\frac{3y}{5}})^2 = 10^2 \quad 3y \geq 0$$

$$\boxed{y \geq 0}$$

check:

$$-8 + \sqrt{\frac{3(500)}{5}} = 2$$

$$-8 + \sqrt{100} = 2$$

$$-8 + 10 = 2$$

✓

When you have just a radical on one side, and just a negative constant on the other, there are no solutions!

Example 3 - Solve a)  $4 + \sqrt{2x-3} = 1$

$$2x-3 \geq 0$$

$$2x \geq 3$$

$$\boxed{x \geq \frac{3}{2}}$$

Check:

$$4 + \sqrt{2(6)-3} = 1$$

$$4 + \sqrt{9} = 1$$

$$4 + 3 = 1$$

$$\times$$

$$\begin{aligned} & \quad \sqrt{2x-3} = -3 \\ & (\sqrt{2x-3})^2 = (-3)^2 \\ & 2x-3 = 9 \\ & 2x = 12 \\ & x = 6 \end{aligned}$$

b)  $-2\sqrt{x-5} = \frac{16}{-2}$

$$\sqrt{x-5} = -8$$

NO SOLUTIONS

Restrict:  $\boxed{x \geq 5}$

Example 4 - Solve

a)  $\sqrt{10x-7} = 3\sqrt{x}$

$$10x-7 \geq 0 \quad x \geq 0$$

$$x \geq \frac{7}{10}$$

so  $\boxed{x \geq \frac{7}{10}}$

$$(\sqrt{10x-7})^2 = (3\sqrt{x})^2$$

$$10x-7 = 9x$$

$x = 7$

Check:

$$\sqrt{10(7)-7} = 3\sqrt{7}$$

$$\sqrt{63} = 3\sqrt{7}$$

$$\sqrt{49} = 3\sqrt{7}$$

$$3\sqrt{7} = 3\sqrt{7}$$

$\checkmark$

b)  $2\sqrt{x} = \sqrt{7x+6}$

$$x \geq 0 \quad 7x+6 \geq 0$$

$$x \geq -\frac{6}{7}$$

so  $\boxed{x \geq 0}$

$$(2\sqrt{x})^2 = (\sqrt{7x+6})^2$$

$$4x = 7x+6$$

$$-3x = 6$$

$$x = -2$$

does not satisfy restriction!

Check:

$$2\sqrt{-2} = \sqrt{7(-2)+6}$$

$$2\sqrt{-2} = \sqrt{-8}$$

$\times$

NO SOLUTIONS