1.1 Factoring x^2+bx+c

Consider the product: $(x + a)(x + b) = x^2 + bx + ax + ab = x^2 + (b + a)x + ab$

This example shows:

1. The product of $(x + a)(x + b)$ is a trinomial.
2. The first term, x^2, is the product of x and x.
3. The coefficient of the middle term is the sum of a and b.
4. The last term is the product of a and b.

These four points lead to a general rule for factoring polynomials of the type $x^2 + bx + c$. When factoring $x^2 + bx + c$, look for two factors of c, that multiply to the coefficient of the last term, and add to the coefficient of the middle term.

Example 1 Factor $x^2 + 7x + 12$

► *Solution*: Since many pairs of integers have a sum of 7, start by finding the pairs of integers with a product of 12.

Integers that multiply to 12: $1 \times 12,\ 2 \times 6,\ 3 \times 4,\ (-1) \times (-12),\ (-2) \times (-6),\ (-3) \times (-4)$

Of these six pairs, only 3 and 4 add to 7.

Therefore $x^2 + 7x + 12 = (x + 3)(x + 4)$.

Checking the answer by FOIL: $(x + 3)(x + 4) = x^2 + 4x + 3x + 12 = x^2 + 7x + 12$.

Example 2 Factor $x^2 - 7x + 6$

► *Solution*: Integers that multiply to 6: $1 \times 6,\ 2 \times 3,\ (-1) \times (-6),\ (-2) \times (-3)$

Of these four pairs, only -1 and -6 have a sum of -7.

Therefore $x^2 - 7x + 6 = (x - 1)(x - 6)$.

Checking the answer by FOIL: $(x - 1)(x - 6) = x^2 - 6x - x + 6 = x^2 - 7x + 6$.

Example 3 Factor $x^2 + 3x - 10$

► *Solution*: Integers that multiply to -10: $1\times(-10)$, $(-1)\times 10$, $2\times(-5)$, $(-2)\times 5$

Of these four pairs, only -2 and 5 have a sum of 3.

Therefore $x^2 + 3x - 10 = (x-2)(x+5)$.

Checking the answer by FOIL gives: $(x-2)(x+5) = x^2 + 5x - 2x - 10 = x^2 + 3x - 10$.

Example 4 Factor $x^2 + 8 - 6x$

► *Solution*: First arrange the trinomial in descending order of powers: $x^2 - 6x + 8$.

Integers that multiply to 8: 1×8, 2×4, $(-1)\times(-8)$, $(-2)\times(-4)$

(-2) and (-4) add to (-6) Therefore
$$\begin{aligned} x^2 + 8 - 6x &= x^2 - 6x + 8 \\ &= (x-4)(x-2) \end{aligned}$$

Example 5 Factor $x^2 - 2x + 4$

► *Solution*: Integers that multiply to 4: 1×4, 2×2, $(-1)\times(-4)$, $(-2)\times(-2)$

None of these pairs add to -2, therefore this polynomial cannot be factored further.

Example 6 Factor $x^2 + 4xy - 21y^2$

► *Solution*: Think of $-21y^2$ as the constant term.

$7y$ and $-3y$ multiply to $-21y^2$ and add to $4y$.

Therefore $x^2 + 4xy - 21y^2 = (x+7y)(x-3y)$.

Example 7 Factor $5x^2 + 35x + 60$

► *Solution*: Always look for a common factor first. The largest common factor is 5.

Therefore
$$\begin{aligned} 5x^2 + 35x + 60 &= 5(x^2 + 7x + 12) \\ &= 5(x+3)(x+4) \end{aligned}$$

Example 8 Factor $-x^2 + 5x + 6$

► *Solution*: First factor out -1, so that the coefficient of x^2 becomes $+1$. This gives $-(x^2 - 5x - 6)$.

Now factor the polynomial $x^2 - 5x - 6$.

-6 and 1 multiply to -6, and add to -5.

Therefore $-x^2 + 5x + 6 = -(x^2 - 5x - 6)$
$= -(x - 6)(x + 1)$

Example 9 Factor $-3x^4 - 18x^3 - 27x^2$

► *Solution*: Always look for a common factor first. The largest common factor is $-3x^2$.

Therefore $-3x^4 - 18x^3 - 27x^2 = -3x^2(x^2 + 6x + 9)$
$= -3x^2(x + 3)(x + 3)$
$= -3x^2(x + 3)^2$

Summary for Factoring $x^2 + bx + c$

1. Arrange the trinomial in descending order of powers.

2. When the last term, c, is positive, the factors of c are both positive, or both negative. If the middle term, b, is positive, both factors are positive. If the middle term is negative, both factors are negative.

Example: $x^2 + 7x + 12 = (x + 4)(x + 3)$ ← *The last term is positive, and the middle term is positive, therefore the factors of 12 are both positive.*

Example: $x^2 - 9x + 20 = (x - 5)(x - 4)$ ← *The last term is positive, and the middle term is negative, therefore the factors of 20 are both negative.*

3. When the last term, c, is negative, the factors of c have opposite signs. The sign of the larger numeric value is also the sign of the coefficient of the middle term.

Example: $x^2 - x - 6 = (x - 3)(x + 2)$ ← *The last term is negative, therefore the signs of the factors of -6 are opposites of each other. Since the middle term of the trinomial is negative, the larger numeric value has a negative sign.*

Example: $x^2 + 2x - 15 = (x + 5)(x - 3)$ ← *The last term is negative, therefore the signs of the factors of -15 are opposite of each other. Since the middle term of the trinomial is positive, the larger numeric value has a positive sign.*

Factoring Difference of Squares $a^2x^2 - b^2y^2,\ a \neq 0, b \neq 0$

The product of the sum and difference of two terms is $(a + b)(a - b) = a^2 - b^2$. Reversing this expression allows the difference of squares to be factored.

> **The Difference of Squares**
>
> $a^2 - b^2 = (a + b)(a - b)$

Example 10 Factor.

a) $x^2 - 16$

b) $x^4 - 4y^2$

c) $x^4 - 16$

d) $16x^2 - 4$

e) $(x^2 - 6x + 9) - y^2$

f) $(16x^2 + 24xy + 9y^2) - (4a^2 - 4ab + b^2)$

► *Solution*:

a) $x^2 - 16 = (x - 4)(x + 4)$

b) $x^4 - 4y^2 = (x^2 - 2y)(x^2 + 2y)$

c) $$\begin{aligned} x^4 - 16 &= (x^2 - 4)(x^2 + 4) \\ &= (x - 2)(x + 2)(x^2 + 4) \end{aligned}$$

d) $$\begin{aligned} 16x^2 - 4 &= 4(4x^2 - 1) \\ &= 4(2x - 1)(2x + 1) \end{aligned}$$

e) $$\begin{aligned} (x^2 - 6x + 9) - y^2 &= (x - 3)(x - 3) - y^2 \\ &= (x - 3)^2 - y^2 \\ &= (x - 3 - y)(x - 3 + y) \end{aligned}$$

f) $$\begin{aligned} (16x^2 + 24xy + 9y^2) - (4a^2 - 4ab + b^2) &= (4x + 3y)(4x + 3y) - (2a - b)(2a - b) \\ &= (4x + 3y)^2 - (2a - b)^2 \\ &= [(4x + 3y) - (2a - b)][(4x + 3y) + (2a - b)] \\ &= (4x + 3y - 2a + b)(4x + 3y + 2a - b) \end{aligned}$$

1.1 Exercise Set

1. Complete the sentences using "positive" or "negative". Assume $x^2 + bx + c$ can be factored into $(x + m)(x + n)$.

a) If b and c are positive, then m is ___________ and n is ___________ .

b) If b is negative and c is positive, then m is ___________ and n is ___________ .

c) If c is negative, then m is ___________ and n is ___________, or m is ___________ and n is ___________ .

d) If m is positive and n is positive, then b is ___________ and c is ___________ .

e) If m is positive and n is negative, then c is ___________, and b might be ___________ or ___________ .

f) If b, c and m are negative, then n is ___________ .

g) If b and c are negative, and m is positive, then n is ___________ .

2. Fill in the blanks to make the statement true.

a) Two positive factors of 6 are _____ , _____ and _____ , _____ .

b) Two negative factors of 9 are _____ , _____ and _____ , _____ .

c) Four factors of 4 are _____ , _____ ; _____ , _____ ; _____ , _____ and _____ , _____ .

d) Six factors of -12 are _____ , _____ ; _____ , _____ ; _____ , _____ ; _____ , _____ ; _____ , _____

and _____ , _____ .

3. Give four examples for b so that the following trinomials can be factored.

a) $x^2 + bx + 6$ _____ , _____ , _____ , _____

b) $x^2 + bx + 4$ _____ , _____ , _____ , _____

c) $x^2 + bx - 8$ _____ , _____ , _____ , _____

d) $x^2 + bx - 6$ _____ , _____ , _____ , _____

4. Give positive and negative integer examples for c so that the following trinomials can be factored.

a) $x^2 + 6x + c$ _____ , _____ , _____ , _____

b) $x^2 - 4x + c$ _____ , _____ , _____ , _____

c) $x^2 + x + c$ _____ , _____ , _____ , _____

d) $x^2 - 5x + c$ _____ , _____ , _____ , _____

5. Complete the factoring.

a) $x^2 + 8x + 15 = (x + 5)(\qquad)$

b) $x^2 - 8x + 15 = (x - 5)(\qquad)$

c) $x^2 + 15x + 44 = (x + 11)(\qquad)$

d) $x^2 - 6x + 9 = (x - 3)(\qquad)$

e) $y^2 + 11y + 28 = (y + 4)(\qquad)$

f) $y^2 - 11y + 30 = (y - 5)(\qquad)$

g) $z^2 + z - 6 = (z + 3)(\qquad)$

h) $z^2 - z - 6 = (z - 3)(\qquad)$

6. Factor.

a) $a^2 + 9a + 8$

b) $b^2 + 16b + 15$

c) $c^2 + 10c + 24$

d) $d^2 + 7d + 10$

e) $x^2 - 18x + 72$

f) $y^2 - 20y + 91$

g) $z^2 - 13z + 36$

h) $u^2 - 4u + 4$

i) $l^2 + 7l - 30$

j) $m^2 + 4m - 12$

7. Factor completely.

a) $3x^2 + 15x + 12$

b) $4y^2 + 20y + 24$

c) $9z^2 + 27z + 18$

d) $2u^2 - 8u + 6$

e) $-5x^2 + 25x - 20$

f) $-2y^2 + 58y - 200$

g) $-3z^2 + 3z + 18$

h) $-4u^2 - 28u + 120$

i) $-x^2 - 6x + 27$

j) $-x^2 + 20x + 44$

k) $-x^2 + 7x + 44$

l) $-x^2 + 6x - 9$

8. Factor completely.

a) $x^3 + 8x^2 - 20x$

b) $x^2 - 12xy + 36y^2$

c) $x^2y - 11xy - 60y$

d) $-2x^4 - 4x^3 + 30x^2$

e) $-3x^4 - 15x^3 + 6x^2$

f) $16x^3 + 48x^2y + 32xy^2$

g) $-x^3y - x^2y^2 + 6xy^3$

h) $2x^4 - 16x^3y + 32x^2y^2$

i) $-x^3y^2 - 3x^2y^3 + 4xy^4$

j) $x^6 - 11x^5y + 28x^4y^2$

9. Factor completely.

a) $x^2 + \frac{5}{4}x + \frac{3}{8}$

b) $x^2 - x + \frac{2}{9}$

c) $x^2 + \frac{1}{4}x - \frac{1}{8}$

d) $\frac{1}{4}x^3 - x^2 - 8x$

e) $x^{2n} + 7x^n + 12$

f) $x^{6n} - 3x^{3n} + 2$

g) $y^{10n} - y^{5n} - 12$

h) $y^{2n} - 7y^n y^m + 10y^{2m}$

i) $x^{a+2} - x^a$

j) $x^{2a+1} + 2x^{a+1} + x$

10. Rewrite the term as the product of equal factors.

a) x^2 __________

b) $9x^2$ __________

c) $16x^6$ __________

d) $100x^2y^4$ __________

e) $81x^8y^{12}$ __________

f) $64x^4y^2z^6$ __________

g) 144 __________

h) $25x^{10}$ __________

i) $225x^{14}$ __________

j) $121x^2y^4z^6$ __________

11. Factor each binomial completely.

a) $x^2 - 1$

b) $4x^2 - 1$

c) $y^2 - 25$

d) $25y^2 - 9$

e) $4 - 9z^2$

f) $16 - 25z^4$

g) $16x^2 - 9y^2$

h) $25x^4 - 81y^6$

i) $16x^2y^8 - 4$

j) $20x^2 - 5y^2$

k) $x^4 - 1$

l) $x^2 + 1$

m) $(x + 1)^2 - y^2$

n) $4 - (x + 2)^2$

12. Factor completely.

a) $(2a + 5)y^2 + 9(2a + 5)y - 10(2a + 5)$

b) $x^3(a + b) - 6x^2(a + b) + 8x(a + b)$

c) $(x - 2y)^2 - 8a(x - 2y) + 15a^2$

d) $(5x - y)^2 + (10xz - 2yz) - 24z^2$

e) $(x + 4)^2 + 2y(x + 4) + y^2$

f) $(x + 4)^2 + y(x + 4) - 2y^2$

13. Factor completely.

a) $(x^2 + 6x + 9) - 4y^2$

b) $(4x^2 + 4xy + y^2) - 9z^2$

c) $x^8 - 1$

d) $(x^6 - 4x^3y^3 + 4y^6) - (a^4 + 6a^2b^2 + 9b^4)$

e) $4^{2m}x^{2m} - 9^{2n}y^{2n}$

f) $25^{4x}y^{6x} - 16^{6x}z^{4x}$

14. Factor completely.

a) $(x^2 + 6xy + 9y^2) - 9(x^2 + 4xy + 4y^2)$

b) $(4a^2 - 9y^2)a^2 - (4a^2 - 9y^2)b^2$

c) $x^{-2} - 4x^{-4}$

d) $2x^{-2} - 7x^{-3} + 3x^{-4}$

e) $\dfrac{x^4}{81} - 1$

f) $x - 1 = (\qquad\quad)(\qquad\quad)$

$\qquad\quad = (\qquad\quad)(\qquad\quad)(\qquad\quad)$

1.2 Factoring ax^2+bx+c

Method 1: The FOIL Method

This method simply reverses the FOIL method for multiplying polynomials. That is, when factoring $ax^2 + bx + c$, we find factor combinations of a and c so that one factor combination adds to b.

Steps:
1. Factor out the greatest common factor, if it exists.

2. Find two first terms whose product is ax^2.

   $(\_\_ x + \quad)(\_\_ x + \quad) = ax^2 + bx + c$

3. Find two last terms whose product is c.

   $(\_\_ \ x + \_\_)(\_\_ \ x + \_\_) = ax^2 + bx + c$

4. Find the outer and inner product for which the sum is bx.

   $(\_\_ x + \_\_)(\_\_ x + \_\_) = ax^2 + bx + c$

Example 1 Factor $2x^2 + 7x - 4$

► *Solution*:
1. There are no common factors of $2x^2$, $7x$, and -4.
2. The factors whose product is $2x^2$ are $(2x)(x)$.

   $(2x + \quad)(x + \quad)$

3. The factors whose product is -4 are:

   $(\quad + 1)(\quad - 4)$, $(\quad - 1)(\quad + 4)$, $(\quad + 2)(\quad - 2)$, $(\quad - 2)(\quad + 2)$

4. Check outer and inner products

   $(2x + 1)(x - 4) \rightarrow -8x + x = -7x$ ← wrong middle term
   $(2x - 1)(x + 4) \rightarrow 8x - x = 7x$ **← correct middle term**
   $(2x + 2)(x - 2) \rightarrow 2(x + 1)(\quad)$ ← reject, implies $2x^2 + 7x - 4 = 2(\quad)(\quad)$
   $(2x - 2)(x + 2) \rightarrow 2(x - 1)(\quad)$ ← reject, implies $2x^2 + 7x - 4 = 2(\quad)(\quad)$

   Therefore $2x^2 + 7x - 4 = (2x - 1)(x + 4)$.

Example 2 Factor $-8x^3 + 10x^2 + 12x$

► *Solution*: Factor out the greatest common factor $-2x$.

$-2x(4x^2 - 5x - 6)$

Now factor $4x^2 - 5x - 6$

Find factors whose product is $4x^2$.

$(x + \quad)(4x + \quad)$ or $(2x + \quad)(2x + \quad)$

Find factors whose product is -6.

$(\quad + 6)(\quad - 1)$, $(\quad - 6)(\quad + 1)$, $(\quad + 2)(\quad - 3)$, $(\quad - 2)(\quad + 3)$

Check outer and inner products

$(x + 6)(4x - 1) \rightarrow -x + 24x = 23x$ ← wrong middle term
$(x - 6)(4x + 1) \rightarrow x - 24x = -23x$ ← wrong middle term
$(x + 2)(4x - 3) \rightarrow -3x + 8x = 5x$ ← wrong middle term
$(x - 2)(4x + 3) \rightarrow 3x - 8x = -5x$ ← **correct middle term**
$(2x + 6)(2x - 1) \rightarrow 2x + 6$ factors, therefore reject
$(2x - 6)(2x + 1) \rightarrow 2x - 6$ factors, therefore reject
$(2x + 2)(2x - 3) \rightarrow 2x + 2$ factors, therefore reject
$(2x - 2)(2x + 3) \rightarrow 2x - 2$ factors, therefore reject

The correct factorization is $(x - 2)(4x + 3)$.

Therefore $-8x^3 + 10x^2 + 12x = -2x(x - 2)(4x + 3)$.

Note 1: A common mistake is to forget to include the common factor $(-2x)$ in the final answer.

Note 2: In example 2, all eight possibilities were listed. When actually doing the problem, you stop as soon as the correct factorization is found. The point of showing all eight possibilities is to see that only the signs of the middle term change when the signs are reversed. So it could be said that only four unique choices exist.

Method 2: The ac -Method

Steps:
1. Factor out a common factor, if it exists.
2. Change $ax^2 + bx + c$ to $x^2 + bx + ac$.
3. Find two integers whose product is ac, and whose sum is b.
4. Divide each constant factor by a, and simplify.
5. If a fraction remains, the denominator becomes the coefficient of the x term in that binomial.

Note: When using this method, make sure to take out a common factor if possible, otherwise it will be eliminated in step 4.

Example 3 Factor $2x^2 + 7x - 4$

► *Solution*: $2x^2 + 7x - 4$ is first changed to $x^2 + 7x - 8$

$(x + 8)(x - 1)$ ← *Factor*

$\left(x + \frac{8}{2}\right)\left(x - \frac{1}{2}\right)$ ← *Divide each constant by 2, the original coefficient of* $2x^2$

$(x + 4)\left(x - \frac{1}{2}\right)$ ← *Simplify*

$(x + 4)(2x - 1)$ ← *If there is a fraction left after the division, the denominator becomes the coefficient of x*

Check: $(x + 4)(2x - 1) = 2x^2 + 7x - 4$

Example 4 Factor $12x^2 - 5x - 2$

► *Solution*: $12x^2 - 5x - 2$ is first changed to $x^2 - 5x - 24$

$(x - 8)(x + 3)$ ← *Factor*

$\left(x - \frac{8}{12}\right)\left(x + \frac{3}{12}\right)$ ← *Divide each constant by 12, the coefficient of* $12x^2$

$\left(x - \frac{2}{3}\right)\left(x + \frac{1}{4}\right)$ ← *Simplify*

$(3x - 2)(4x + 1)$ ← *If there is a fraction left after the division, the denominator becomes the coefficient of x*

Check: $(3x - 2)(4x + 1) = 12x^2 - 5x - 2$

Factoring Perfect Square Trinomials $ax^2 + bx + c$

A trinomial that is a square of a binomial is called a perfect square trinomial.

For trinomial $ax^2 + bx + c$ to be a perfect square:

a) The last term must be positive, and a perfect square.
b) The first term must be a perfect square.
c) The coefficient of the middle term is the square root of the first term, multiplied by the square root of the last term, then doubled. This could be positive or negative.

Factoring Perfect Square Trinomials

$a^2 + 2ab + b^2 = (a+b)^2$

$a^2 - 2ab + b^2 = (a-b)^2$

Example 5 Factor.

a) $x^2 + 8x + 16$

b) $4x^2 - 12x + 9$

c) $(x-2y)^2 - 2(x-2y) + 1$

► *Solution*: **a)** $x^2 + 8x + 16 = (x+4)(x+4) = (x+4)^2$

b) $4x^2 - 12x + 9 = (2x-3)(2x-3) = (2x-3)^2$

c) $(x-2y)^2 - 2(x-2y) + 1 = (x-2y-1)(x-2y-1) = (x-2y-1)^2$

Example 6 Find all integers k, so that the trinomial is a perfect square.

a) $x^2 + 6x + k$

b) $9x^2 - kx + 4$

c) $kx^2 + 12x + 9$

► *Solution*: **a)** A number must be found that when multiplied by itself results in k and when added to itself is 6. Therefore the perfect square is $(x+3)^2$ and $k = 9$.

b) Since the coefficients of the first and last terms are perfect squares
$9x^2 - kx + 4 = (3x-2)(3x-2)$ or $(3x+2)(3x+2)$. Therefore $k = 12$ or -12

c) Since the trinomial is a perfect square, $kx^2 + 12x + 9 = (\_x + 3)(\_x + 3)$.
The coefficient of x is 2 since it produces a middle term of 12. Therefore $k = 4$.

1.2 Exercise Set

1. Complete the following factoring.

a) $6x^2 + 11x + 4 = (2x + 1)(\qquad)$

b) $8x^2 + 18x + 9 = (4x + 3)(\qquad)$

c) $2x^2 + 11x + 15 = (2x + 5)(\qquad)$

d) $8x^2 - 14x + 3 = (4x - 1)(\qquad)$

e) $12x^2 - 20x + 3 = (2x - 3)(\qquad)$

f) $9x^2 - 6x + 1 = (3x - 1)(\qquad)$

g) $9x^2 + 3x - 2 = (3x - 1)(\qquad)$

h) $2x^2 - x - 15 = (x - 3)(\qquad)$

2. Factor.

a) $2x^2 + 13x + 15$

b) $3x^2 + 8x + 4$

c) $10x^2 + 17x + 3$

d) $8y^2 - 18y + 9$

e) $21y^2 - 41y + 10$

f) $2y^2 - 7y + 5$

g) $20z^2 - 27z - 8$

h) $3z^2 - 20z - 63$

i) $8z^2 + z - 9$

j) $15z^2 - 16z - 15$

k) $6a^2 + 17a - 3$

l) $6a^2 - a - 1$

m) $4a^2 - 7a + 3$

n) $4a^2 + 4a - 3$

3. Factor.

a) $-3x^2 - x + 4$

b) $-2x^2 - 5xy - 2y^2$

c) $-5x^2 + 2x + 16$

d) $-3x^2 + 13xy - 4y^2$

e) $-100x^2 + 120xy - 32y^2$

f) $-36x^2 - 96xy - 64y^2$

g) $-20x^2 - 16x - 3$

h) $-6x^2 - 9xy + 42y^2$

i) $-15x^2 + 26xy - 8y^2$

j) $-12x^4 + 22x^2 + 20$

4. Factor.

a) $25x^2(a-1)^3 - 5x(a-1)^3 - 2(a-1)^3$

b) $-3x^2(y+1)^2 - 2x(y+1)^2 + 5(y+1)^2$

c) $1 - 7x - 60x^2$

d) $9 - 10x^2 + x^4$

e) $x^4(1-x)^3 - 20x^2(1-x)^3 + 64(1-x)^3$

f) $18y^2(x+1)^2 - 21y(x+1)^2 - 4(x+1)^2$

g) $15a^2(a-2)^2 - 34ab(a-2)^2 - 16b^2(a-2)^2$

h) $4x^2(2-z)^5 + 20xy(2-z)^5 + 25y^2(2-z)^5$

5. Are the following perfect square trinomials?

a) $x^2 + 4x + 4$ y / n

b) $x^2 + 8x + 4$ y / n

c) $x^2 + 6x + 9$ y / n

d) $x^2 + 8x + 9$ y / n

e) $4x^2 - 10x + 9$ y / n

f) $4x^2 - 12x + 9$ y / n

g) $x^4 + 10x^2 + 25$ y / n

h) $x^4 - 2x^2 + 1$ y / n

i) $36x^2 - 12xy + y^2$ y / n

j) $25x^2 - 20xy + 4y^2$ y / n

6. Factor each trinomial completely.

a) $x^2 + 10x + 25$

b) $x^2 + 8x + 16$

c) $y^2 - 12y + 36$

d) $y^4 - 6y^2 + 9$

e) $2z^2 - 28z + 98$

f) $3z^2 - 30z + 75$

g) $x^3 - 16x^2 + 64x$

h) $9x^2 - 24xy + 16y^2$

i) $-50a^2 + 40ab - 8b^2$

j) $-9x^2 - 24xy - 16y^2$

7. Find all integers k which make the trinomial a perfect square.

a) $x^2 + 8x + k$

b) $y^2 - 6y + k$

c) $4z^2 + kz + 9$

d) $9x^2 - kxy + 16y^2$

e) $ky^2 + 24y + 9$

f) $kz^2 - 24z + 9$

g) $64x^2 + 112x + k$

h) $25y^2 - 40y + k$

i) $kx^2 - 24x + 16$

j) $9x^2 + kxy + 25y^2$

8. Factor each polynomial, assuming that m and n are natural numbers.

a) $4x^{2m} - 20x^m y^n + 25y^{2n}$

b) $10x^{2m} - 4x^m y^n - 6y^{2n}$

c) $x^{2m} - 4y^{2n}$

d) $x^{4m} - y^{4n}$

e) $-4x^{4m} - 6x^{2m}y^{2m} + 4y^{4m}$

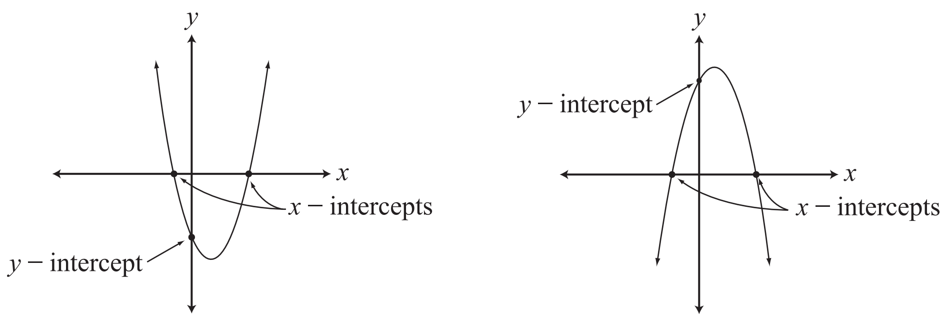
f) $-6x^{8m} + 17x^{4m}y^{4n} - 10y^{8n}$

1.3 Properties of Quadratic Functions

Quadratic functions are not just found in a classroom. Tossing a ball in the air is a quadratic function, so is the flow rate of water in a pipe, the arc of the supporting cables of a suspension bridge and the shape of an astronomical telescope lens.

A quadratic function is a function that can be written in the **general form** $f(x) = ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$. The graph of a quadratic function is called a **parabola**.

If the equation of the parabola is written in the form $y = f(x) = ax^2 + bx + c$, the parabola opens **upward** if $a > 0$ and **downward** if $a < 0$.



$a > 0$ $a < 0$

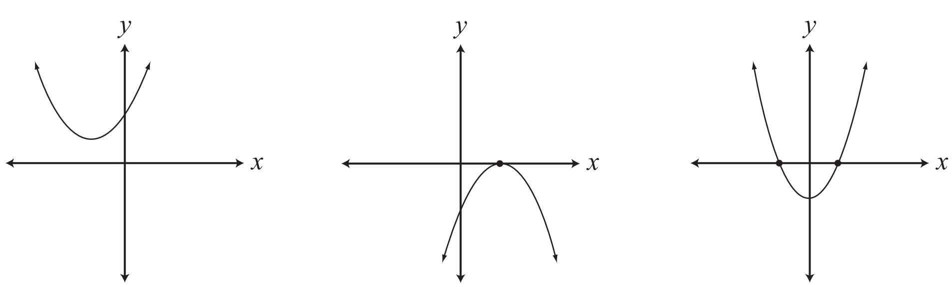
The *y*-intercept

The y-intercept is the point where the parabola intersects the y-axis. It is found by letting $x = 0$ in the parabola $y = f(x) = ax^2 + bx + c$.

There is always just one y-intercept, the point $(0, c)$.

The *x*-intercept

The x-intercepts are the points where the parabola intersects the x-axis. There can be 0, 1, or 2 x-intercepts, depending on the quadratic function.

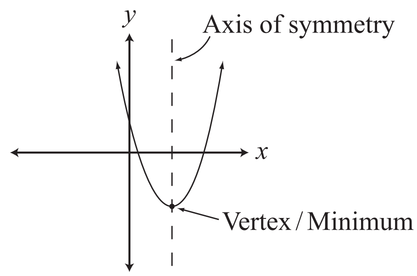


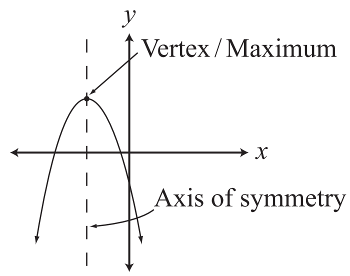
no x-intercept one x-intercept two x-intercepts

The Vertex and Axis of Symmetry

The maximum or minimum point of a parabola is called the **vertex**. The y-value of the vertex is the maximum or minimum value of the parabola.

The vertical line that passes through the vertex of the parabola is called the **axis of symmetry**. If a parabola is folded along the axis of symmetry, the two sides of the graph coincide. The equation of the axis of symmetry is $x = n$, where n is the x-coordinate of the vertex.



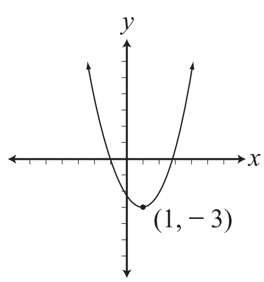


Domain and Range

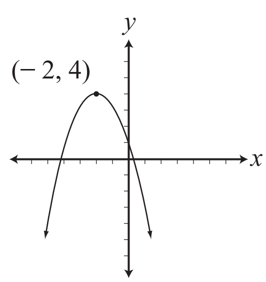
The domain of the quadratic function $y = f(x) = ax^2 + bx + c$ is all real numbers, since there is no limitation on what the x value can be.

The vertex of the parabola helps find the range for the quadratic function.

If the vertex is (x, n), then the range is: $y \geq n$, when the parabola opens upward.
$y \leq n$, when the parabola opens downward.



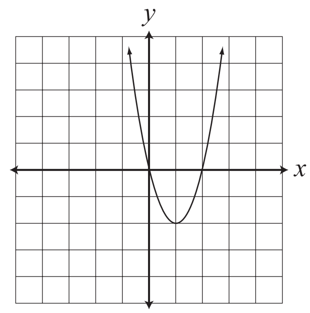
Vertex: $(1, -3)$
Domain: All real numbers
Range: $y \geq -3$



Vertex: $(-2, 4)$
Domain: All real numbers
Range: $y \leq 4$

Example 1 Using the graph of the parabola $y = 2x^2 - 4x$, determine the:

a) Vertex
b) Axis of symmetry
c) y-intercept
d) x-intercepts
e) Domain of f
f) Range of f
g) Minimum value



► *Solution*: **a)** $(1, -2)$
b) $x = 1$
c) $(0, 0)$
d) $(0, 0)$ and $(2, 0)$
e) All real numbers
f) $y \geq -2$
g) -2

Example 2 Find the x-intercept(s), y-intercept, and axis of symmetry of the function $f(x) = x^2 + 2x - 15$.

► *Solution*: x-intercept has $y = 0$: $f(x) = x^2 + 2x - 15 = 0$

$$(x+5)(x-3) = 0$$

$$x = -5, 3; \ x\text{-intercepts } (-5, 0), (3, 0)$$

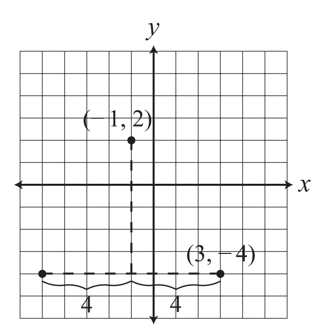
y-intercept has $x = 0$: $f(0) = 0^2 + 2(0) - 15$

$$= -15; \ y\text{-intercept } (0, -15)$$

The axis of symmetry is the line through the midpoint of x-intercepts -5 and 3: $x = \dfrac{-5+3}{2} = -1$

Example 3 Find a point on a quadratic function that has vertex $(-1, 2)$ and passes through the point $(3, -4)$.

► *Solution*: Because of the symmetry of a quadratic function, when two points on the function have the same y-value, the x-values of those points must be equal distance from the x-value of the vertex.



The distance from -1 to 3 is 4 units to the right.

4 units to the left of -1 is -5.

Therefore another point on the function is $(-5, -4)$.

1.3 Exercise Set

1. Fill in the blanks.

a) A function of the form $y = f(x) = ax^2 + bx + c$ is called a _____________ function.

b) A function of the form $y = f(x) = ax^2 + bx + c$ has a graph called a _____________ .

c) The lowest point on a graph that opens upward is the _____________ of the parabola.

d) The vertex is the highest point on a parabola that opens _____________ .

e) The line that divides a parabola into two separate halves is called _____________ .

f) A parabola is a polynomial of degree _____________ .

g) A parabola $y = f(x) = ax^2 + bx + c$ with $a > 0$, is a parabola that opens _____________ .

h) A parabola $y = f(x) = ax^2 + bx + c$ with $a < 0$, is a parabola that opens _____________ .

i) The point on a parabola $y = f(x) = ax^2 + bx + c$ where $f(0) = c$, is the _____________ of the parabola.

j) The point on a parabola $y = f(x) = ax^2 + bx + c$ where $f(x) = 0$, is the _____________ of the parabola.

2. Which functions are quadratic functions?

a) $y = x^2 + 4$ **b)** $y = 3x - 2$ **c)** $y = x^2 + \sqrt{2}\,x$ **d)** $y = x^2 + 2\sqrt{x}$ **e)** $y = x^2 + \frac{x}{3}$ **f)** $y = x^2 + \frac{3}{x}$

3. Find the x-intercept(s) (if possible), y-intercept , and axis of symmetry of the quadratic function.

a) $y = f(x) = x^2 - 6x + 9$

b) $y = g(x) = x^2 - x - 2$

c) $y = h(x) = 2x^2 + x - 6$

d) $y = j(x) = 9 - x^2$

e) $y = k(x) = -x^2 + 2x + 3$

f) $y = l(x) = x^2 + 1$

g) $y = m(x) = -x^2 - 2x + 8$

h) $y = n(x) = 4 - 4x + x^2$

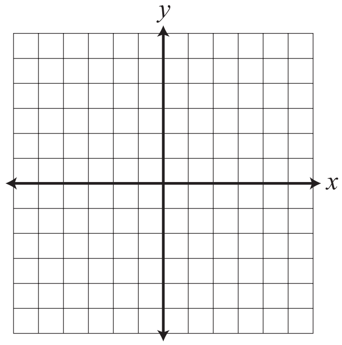
i) $y = p(x) = x^2 - \sqrt{2}\,x$

j) $y = q(x) = -\sqrt{6}\,x^2 - \sqrt{2}\,x$

4. Graph the quadratic function by completing a table of values.

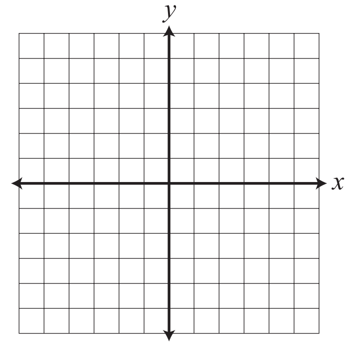
a) $f(x) = x^2 - 4$

x	y
-3	
-2	
-1	
0	
1	
2	
3	



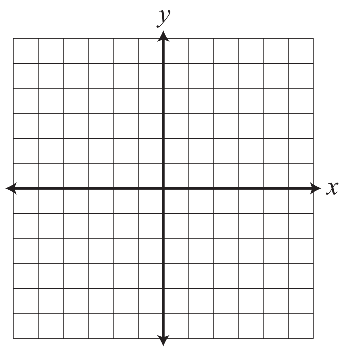
b) $g(x) = -x^2 + 4$

x	y
-3	
-2	
-1	
0	
1	
2	
3	



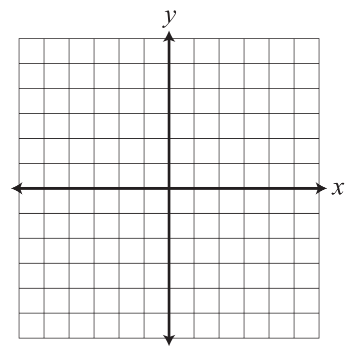
c) $h(x) = x^2 - x$

x	y
-3	
-2	
-1	
0	
1	
2	
3	



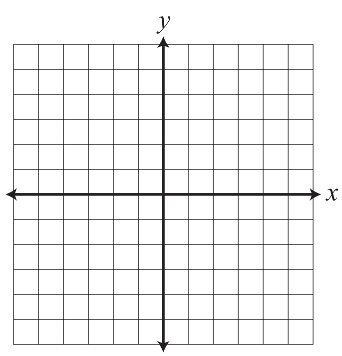
d) $j(x) = -x^2 + x$

x	y
-3	
-2	
-1	
0	
1	
2	
3	



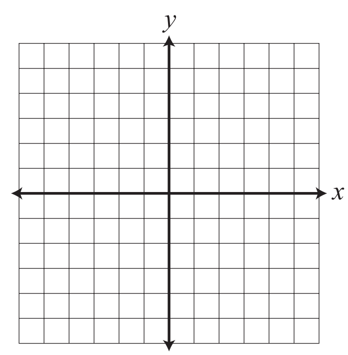
e) $k(x) = \frac{1}{2}x^2 - 6$

x	y
-3	
-2	
-1	
0	
1	
2	
3	



f) $l(x) = -2x^2 + 6$

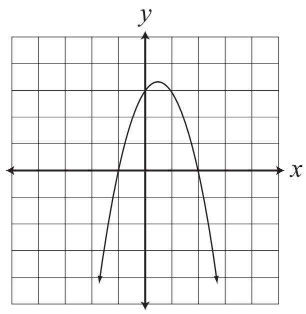
x	y
-3	
-2	
-1	
0	
1	
2	
3	



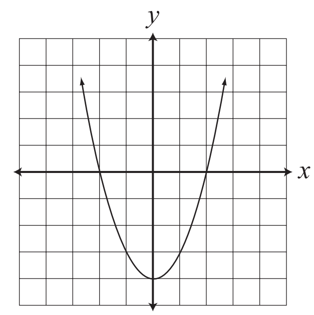
5. Find the following features of the given parabolas.

i) Vertex
ii) Axis of symmetry
iii) x-intercept(s)
iv) y-intercept
v) Domain of f
vi) Range of f
vii) Maximum or minimum value

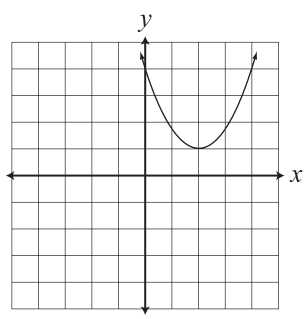
a) *i*) __________
ii) __________
iii) __________
iv) __________
v) __________
vi) __________
vii) __________



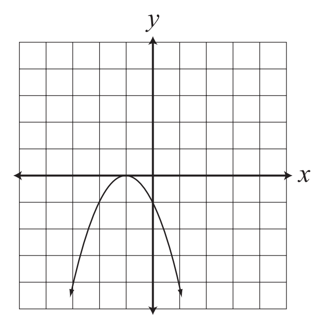
b) *i*) __________
ii) __________
iii) __________
iv) __________
v) __________
vi) __________
vii) __________



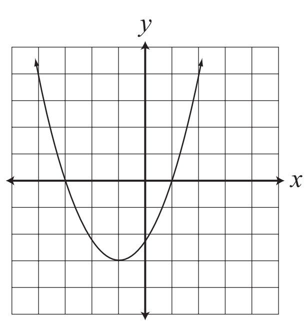
c) *i*) __________
ii) __________
iii) __________
iv) __________
v) __________
vi) __________
vii) __________



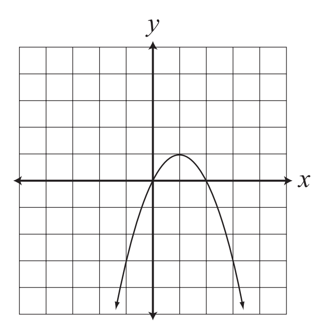
d) *i*) __________
ii) __________
iii) __________
iv) __________
v) __________
vi) __________
vii) __________



e) *i*) __________
ii) __________
iii) __________
iv) __________
v) __________
vi) __________
vii) __________



f) *i*) __________
ii) __________
iii) __________
iv) __________
v) __________
vi) __________
vii) __________



6. Find a third point on the quadratic function that has the indicated vertex, and whose graph passes through the given point.

a) vertex $(3, 2)$, point $(5, 7)$

b) vertex $(-1, 4)$, point $(2, 5)$

c) vertex $(3, 0)$, point $(7, 4)$

d) vertex $(2, -5)$, point $(-3, 2)$

e) vertex $(-4, 2)$, point $(-7, 8)$

f) vertex $(-1, -4)$, point $(3, -6)$

g) vertex (c, d), point $(-3, 1)$

h) vertex $(-3, 1)$, point (c, d)

i) vertex (a, b), point (c, d)

j) vertex (c, d), point (a, b)

7. If the graph of a quadratic function passes through the given points, find the equation of the axis of symmetry. Does the parabola open upward or downward?

a) $(0, 0), (5, 0), (3, 1)$

b) $(-6, 3), (-5, -2), (-2, 3)$

c) $(5, 2), (2, 8), (6, 8)$

d) $(-4, -1), (-8, -3), (0, -1)$

e) $(-7, -5), (-3, 2), (4, 2)$

f) $(-3, 4), (6, 6), (5, 4)$

g) $(a, b), \left(\frac{a}{2}, c\right), (-a, b),\ b > c$

h) $(a, b), \left(\frac{a}{2}, c\right), (-3a, b),\ b < c$

1.4 The Standard Form of a Quadratic Function

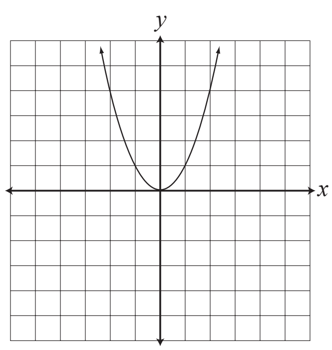
From the equation $f(x) = ax^2 + bx + c$, the direction the parabola opens is determined by the sign of a, and the y-intercept is determined by the value of c. However, the vertex of the parabola is not easily recognized. The **standard form** of a quadratic function, $f(x) = a(x-h)^2 + k$, allows the vertex to be easily identified.

Graphing a Parabola

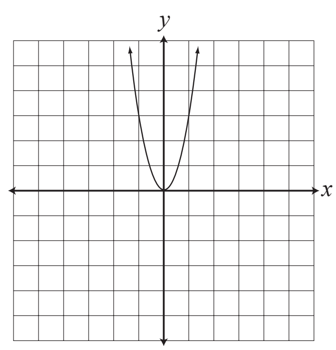
1. Find the vertex and axis of symmetry.
2. Find the y-intercept by evaluating $f(0)$, and the x-intercepts by evaluating $f(x) = 0$.
3. Add additional points if needed.
4. Sketch graph.

Consider the graph of $y = ax^2$:

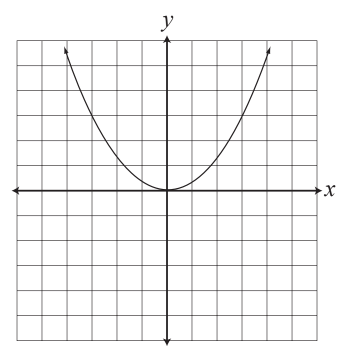
The vertex is at the origin $(0, 0)$.
If $a > 0$, the parabola opens upward and the vertex is a minimum point.
If $a < 0$, the parabola opens downward and the vertex is a maximum point.
In comparison to $y = x^2$, the parabola will be wide if $-1 < a < 1$, and narrow if $a > 1$ or $a < -1$.
The axis of symmetry is $x = 0$.



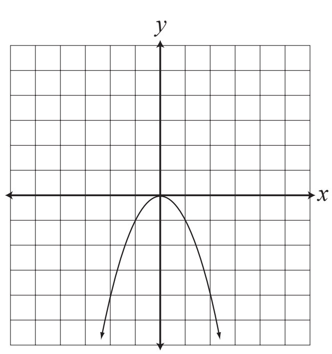
$a = 1$



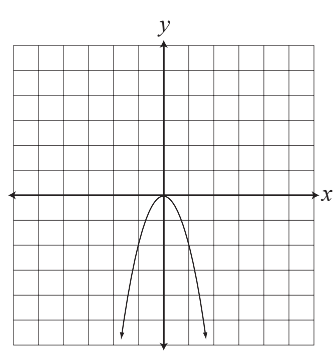
$a = 3$



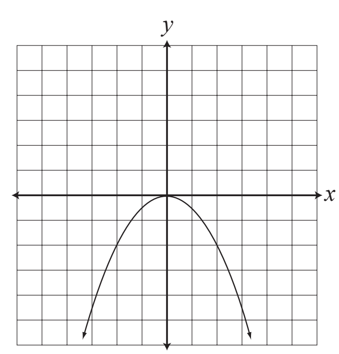
$a = \frac{1}{3}$



$a = -1$



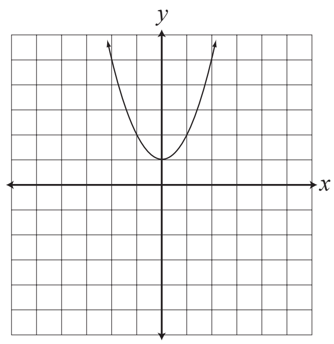
$a = -2$



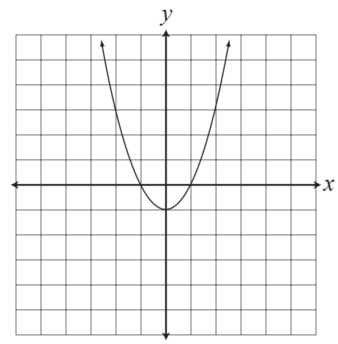
$a = -\frac{1}{2}$

Consider the graph of $y = ax^2 + c$:

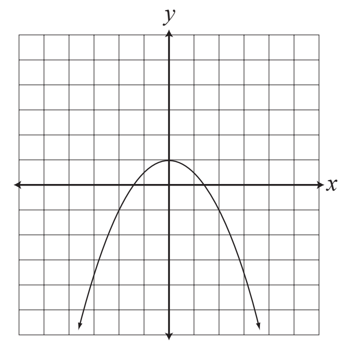
The vertex of the parabola is $(0, c)$.
The axis of symmetry is $x = 0$.
The graph is shifted vertically up if $c > 0$, and vertically down if $c < 0$.



$y = x^2 + 1$



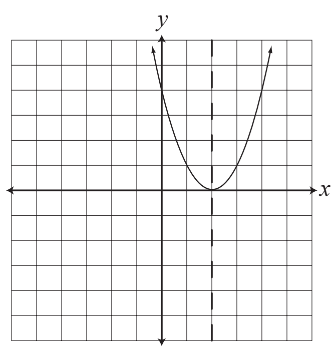
$y = x^2 - 1$



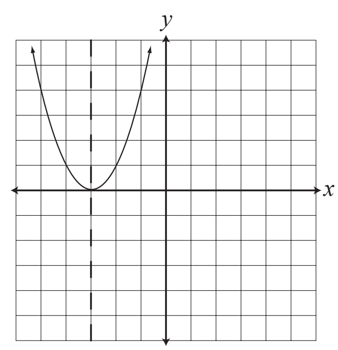
$y = -\frac{1}{2}x^2 + 1$

Consider the graph of $y = a(x - c)^2$:

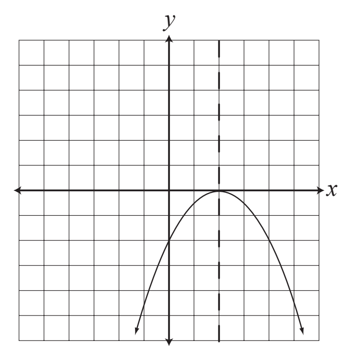
The vertex of the parabola is $(c, 0)$.
The graph shifts horizontally to the right if $c > 0$, and horizontally to the left if $c < 0$.
The axis of symmetry is $x = c$.



$y = (x - 2)^2$



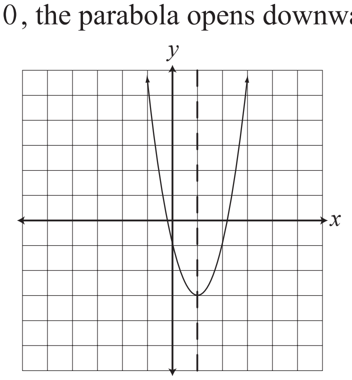
$y = (x + 3)^2$



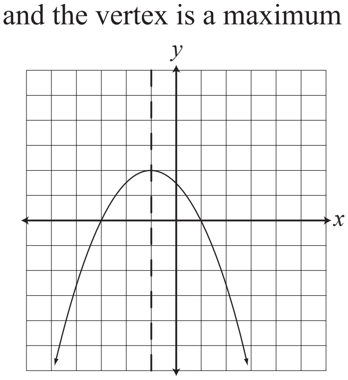
$y = -\frac{1}{2}(x - 2)^2$

Consider the graph of $y = a(x - h)^2 + k$:

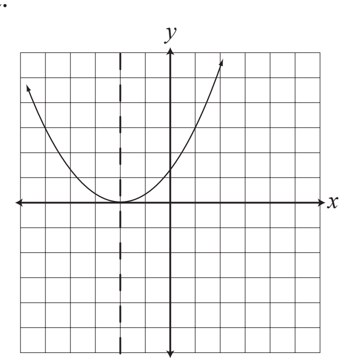
The axis of symmetry is $x = h$.
The vertex of the parabola is (h, k).
If $a > 0$, the parabola opens upward, and the vertex is a minimum point.
If $a < 0$, the parabola opens downward, and the vertex is a maximum point.



$y = 2(x - 1)^2 - 3$



$y = -\frac{1}{2}(x + 1)^2 + 2$



$y = \frac{1}{3}(x + 2)^2$

Example 1 Graph the function $f(x) = -\frac{1}{4}(x-1)^2 + 4$.

► *Solution*: The parabola opens downward because the x^2 coefficient is negative.

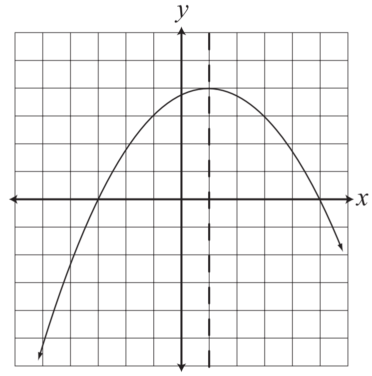
Vertex: $(1, 4)$

Axis of symmetry: $x = 1$

x-intercept: $f(x) = -\frac{1}{4}(x-1)^2 + 4 = 0$

$$(x-1)^2 = 16$$
$$x - 1 = \pm 4$$
$$x = -3, 5$$

x-intercepts are $(-3, 0), (5, 0)$



y-intercept: $f(0) = -\frac{1}{4}(0-1)^2 + 4$

$$= -\frac{1}{4} + 4$$
$$= 3\frac{3}{4}$$

y-intercept is $\left(0, 3\frac{3}{4}\right)$

Domain: all real numbers

Range: $y \leq 4$

Maximum value: 4

Example 2 Graph the function $g(x) = 3(x+1)^2 - 3$.

► *Solution*: The parabola opens upward because the x^2 coefficient is positive.

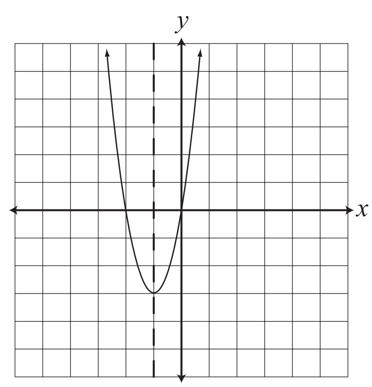
Vertex: $(-1, -3)$

Axis of symmetry: $x = -1$

x-intercept: $f(x) = 3(x+1)^2 - 3 = 0$

$$(x+1)^2 = 1$$
$$x + 1 = \pm 1$$
$$x = -2, 0$$

x-intercepts are $(-2, 0), (0, 0)$



y-intercept: $g(0) = 3(0+1)^2 - 3$

$$= 3 - 3$$
$$= 0$$

y-intercept is $(0, 0)$

Domain: all real numbers

Range: $y \geq -3$

Minimum value: -3

1.4 Exercise Set

1. Match the equation of the quadratic function with its corresponding graph.

a) $f(x) = 2(x-1)^2 - 3$ ______

b) $f(x) = -2(x+1)^2 - 3$ ______

c) $f(x) = -2(x-1)^2 + 3$ ______

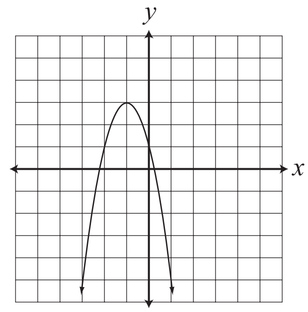
d) $f(x) = -2(x-1)^2 - 3$ ______

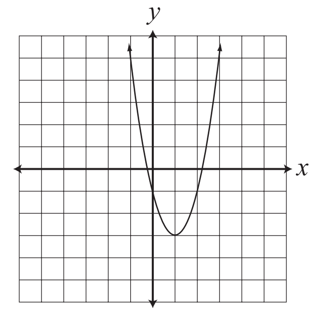
e) $f(x) = 2(x-1)^2 + 3$ ______

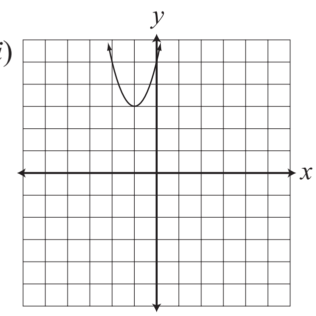
f) $f(x) = -2(x+1)^2 + 3$ ______

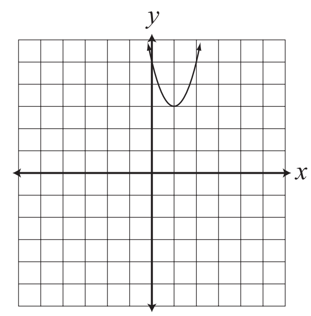
g) $f(x) = 2(x+1)^2 - 3$ ______

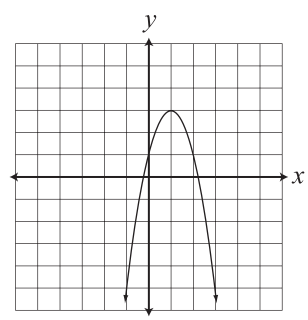
h) $f(x) = 2(x+1)^2 + 3$ ______

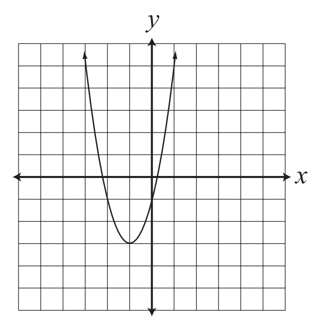
i)


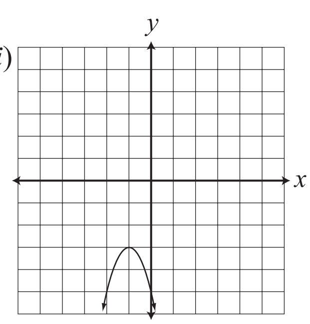
ii)


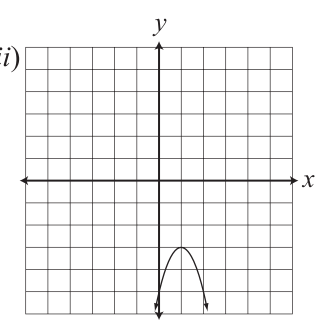
iii)


iv)


v)


vi)


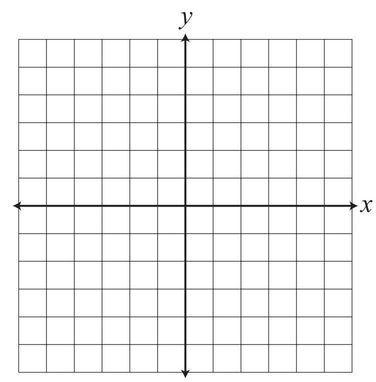
vii)


viii)


2. Graph each function $y = f(x) = a(x-h)^2 + k$, using at least 5 points. Determine the:

i) Vertex
ii) Axis of symmetry
iii) y-intercept
iv) x-intercepts
v) Domain of f
vi) Range of f
vii) Maximum / minimum value

a) $y = -(x-1)^2 + 4$



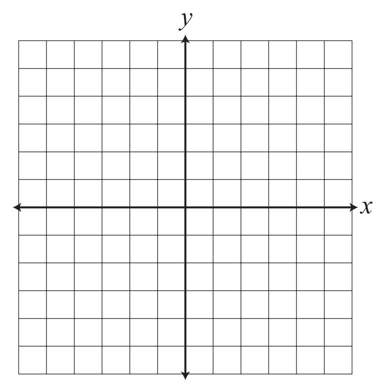
i) *ii*)

iii) *iv*)

v) *vi*)

vii)

b) $y = \frac{1}{4}(x+2)^2$



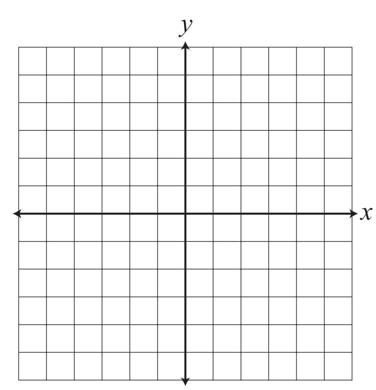
i) *ii*)

iii) *iv*)

v) *vi*)

vii)

c) $y = -\frac{1}{2}x^2 + 6$



i) *ii*)

iii) *iv*)

v) *vi*)

vii)

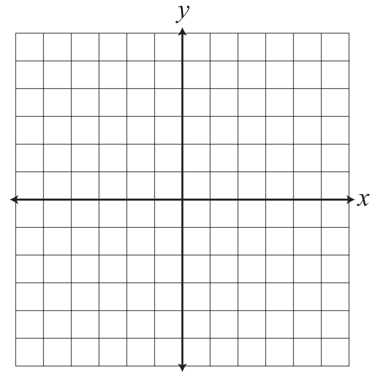
2. **d)** $y = -2(x+1)^2 + 3$

i) *ii*)

iii) *iv*)

v) *vi*)

vii)



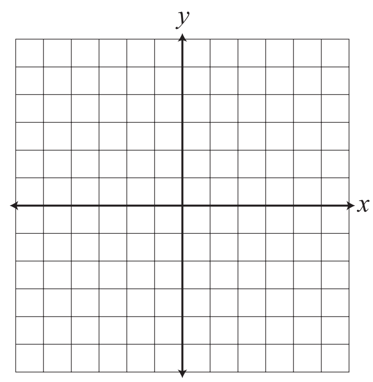
e) $y = \frac{3}{2}(x+2)^2 - 6$

i) *ii*)

iii) *iv*)

v) *vi*)

vii)



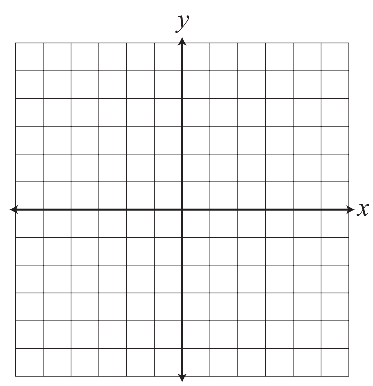
f) $y = -\frac{3}{2}(x-1)^2 + 6$

i) *ii*)

iii) *iv*)

v) *vi*)

vii)



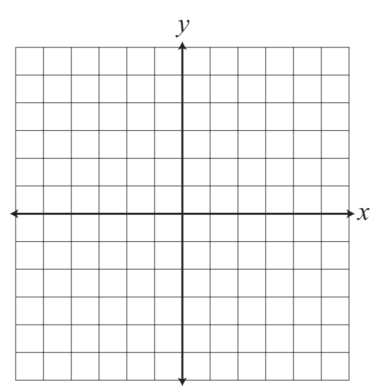
g) $y = \frac{3}{4}(x-2)^2 - 3$

i) *ii*)

iii) *iv*)

v) *vi*)

vii)



3. Find the number of x-intercepts for the quadratic function.

a) $f(x) = -2(x-1)^2 + 1$ ________

b) $g(x) = -2(x-1)^2 - 1$ ________

c) $h(x) = -2(x-1)^2$ ________

d) $j(x) = 2(x+1)^2 + 1$ ________

e) $k(x) = 3x^2 - 1$ ________

f) $l(x) = -3x^2 - 1$ ________

4. Write an equation for a quadratic function with the same shape as $f(x) = \frac{2}{3}x^2$, but with the given point as a vertex.

a) $(0, 2)$

b) $(2, 0)$

c) $(-1, 3)$

d) $(1, -3)$

e) $(4, 2)$

f) $(-2, -4)$

g) $(-a, -b)$

h) $(a - b, c)$

5. Write an equation of a parabola that has the shape $f(x) = \pm 3x^2$, and the given vertex.

a) maximum $(2, 0)$

b) minimum $(0, -1)$

c) maximum $(-1, 4)$

d) minimum $(1, -2)$

e) maximum $(\sqrt{2}, -\sqrt{3})$

f) minimum $(-1 + \sqrt{2}, \sqrt{5})$

6. Write the equation of a quadratic function $h(x)$ that has the same shape as $f(x) = -\frac{1}{2}(x+1)^2 - 3$ and the same vertex as $g(x) = 2(x+2)^2 + 1$.

7. Write the equation of a quadratic function $h(x)$ that has the same shape as $f(x) = \frac{2}{3}(x-1)^2 + 2$ and the same vertex as $g(x) = -(x+3)^2 - 4$.

1.5 Chapter Review

Section 1.1

1. Factor completely.

a) $x^2 - 10x + 24$

b) $x^2 - 4x - 45$

c) $x^4 - 16$

d) $x^2 - xy - 12y^2$

e) $50 - 5y - y^2$

f) $3x^2 - 12$

g) $-x^2 - y^2$

h) $4x^3 - 12x^2y + 8xy^2$

i) $3x^2 - 6x$

j) $x^2y^2 + 7xy + 12$

k) $-2x^6 + 8x^5 - 8x^4$

l) $x^4y - 81y^5$

2. Find all integer values of k such that the trinomial factors.

a) $x^2 + kx + 10$

b) $x^2 + kx - 10$

c) $x^2 + 3x + k,\ -20 < k < 20$

d) $x^2 - 2x + k,\ -20 < k < 20$

Section 1.2

3. Factor completely.

a) $2x^2 + 11x + 12$

b) $6x^2 - 17x - 3$

c) $8x^2 + 18x - 5$

d) $9x^3 + 12x^2 - 45x$

e) $x^3 + 3x^2 - 9x - 27$

f) $-12 - x^2y^2 - 8xy$

g) $27x^2 - 144xy + 192y^2$

h) $6(x-1)^2 + 7y(x-1) - 3y^2$

i) $18x^3y + 3x^2y^2 - 6xy^3$

j) $2x(x-3) + (x-1)(x+2)$

4. Find all integer values of k such that the trinomial factors.

a) $2x^2 + kx - 3$

b) $2x^2 - 5x + k, \; -20 < k < 20$

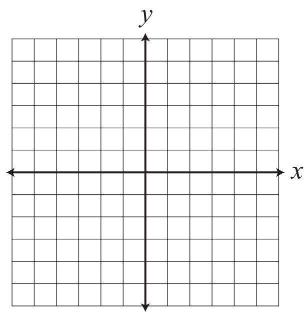
c) $3x^2 + kx - 4$

d) $3x^2 + x + k, \; -20 < k < 20$

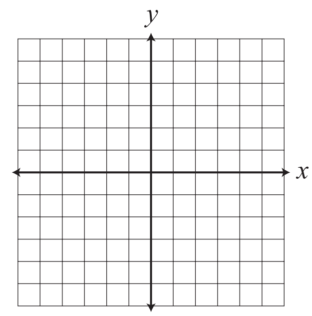
Section 1.3

5. Graph and state the vertex, axis of symmetry, x-intercept(s), y-intercept, domain, range, and maximum/minimum value of the following quadratic functions.

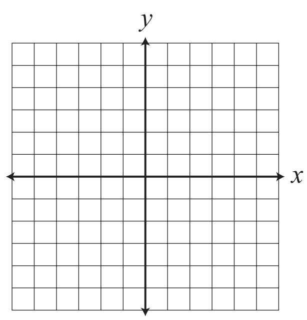
a) $f(x) = \frac{1}{2}x^2 - 2x$



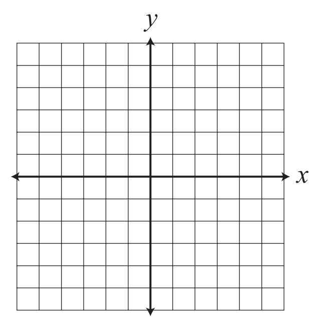
b) $g(x) = -\frac{1}{2}x^2 + 3$



c) $y = -\frac{1}{3}x^2 + 2x - 4$



d) $y = \frac{1}{2}x^2 - x - 4$



6. Find another point on the quadratic function, if the vertex and a point on the parabola are given.

a) vertex $(-2, 4)$, point $(0, 0)$ _________

b) vertex $(0, -4)$, point $(-3, 0)$ _________

c) vertex $(3, 1)$, point $(-3, 4)$ _________

d) vertex $(-5, 3)$, point $(2, -4)$ _________

e) vertex $(-3, -1)$, point $(2, -2)$ _________

f) vertex $(-1, -3)$, point $(-2, 2)$ _________

Section 1.4

7. Determine the direction and distance $g(x)$ is shifted in relation to $f(x)$.

a) $g(x) = f(x) + 3$

b) $g(x) = f(x) - 2$

c) $g(x) = f(x + 4)$

d) $g(x) = f(x - 1)$

8. For the graph of each function, state the vertex, axis of symmetry, x-intercept(s), y-intercept, domain, range, and maximum/minimum value.

a) $y = 2(x - 1)^2 + 3$

b) $y = -(x + 2)^2 + 1$

c) $y + 4 = (x + 1)^2$

d) $y = -2(x + 3)^2 + 8$

9. Write an equation of a parabola that has the shape $f(x) = \pm\frac{1}{2}x^2$ with the given vertex.

a) minimum $(3, -2)$

b) maximum $(1, 4)$

c) minimum $(-1, 5)$

d) maximum $(-3, -1)$