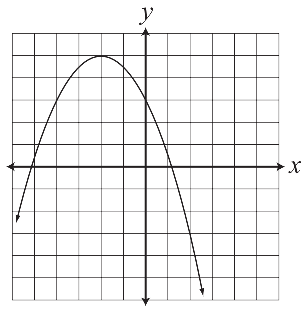
2.1 Finding the Equation of a Parabola

By looking at the sketch of a parabola, the equation for the parabola can be determined.

Consider the parabola:



The vertex of the parabola is $(-2, 5)$, therefore $y = a(x+2)^2 + 5$.

Three other points on the graph are $(0, 3)$, $(-4, 3)$, and $(2, -3)$. To determine a, any one of these points can be substituted into $y = a(x+2)^2 + 5$.

Using $(0, 3)$:

$$\begin{aligned} y &= a(x+2)^2 + 5 \\ 3 &= a(0+2)^2 + 5 \\ 3 &= 4a + 5 \\ 4a &= -2 \\ a &= -\frac{1}{2} \end{aligned}$$

Therefore the equation of the parabola is $y = -\frac{1}{2}(x+2)^2 + 5$.

Note: The points $(-4, 3)$ or $(2, -3)$ will give the same result.

Example 1 Find the equation of a quadratic function whose graph has vertex $(4, 8)$ and x-intercept 6.

► *Solution*: If the vertex is $(4, 8)$ then $f(x) = a(x-4)^2 + 8$.

The x-intercept is the point $(6, 0)$ which is on the graph.

Solving for a:

$$\begin{aligned} f(6) = a(6-4)^2 + 8 &= 0 \\ 4a + 8 &= 0 \\ 4a &= -8 \\ a &= -2 \end{aligned}$$

Therefore $f(x) = -2(x-4)^2 + 8$.

Example 2 Find the equation of a quadratic function with points $(-1, 0), \left(0, \frac{3}{2}\right), (3, 0)$.

► *Solution*: Let $f(x) = a(x-h)^2 + k$.

$(-1, 0)$ and $(3, 0)$ are symmetric because they have the same y-value. Therefore the axis of symmetry must be a vertical line equidistant from the two points. Its equation is $x = \frac{-1+3}{2} = 1$

Therefore $f(x) = a(x-1)^2 + k$.

Substituting $(3, 0)$ into the equation:

$$\begin{aligned} f(3) = a(3-1)^2 + k &= 0 \\ 4a + k &= 0 \\ k &= -4a \end{aligned}$$

Therefore $f(x) = a(x-1)^2 - 4a$.

Using $\left(0, \frac{3}{2}\right)$:

$$\begin{aligned} f(0) = a(0-1)^2 - 4a &= \frac{3}{2} \\ a - 4a &= \frac{3}{2} \\ -3a &= \frac{3}{2} \\ a &= -\frac{1}{2} \end{aligned}$$

Therefore $k = -2$, and $f(x) = -\frac{1}{2}(x-1)^2 + 2$.

Example 3 Find an equation of a quadratic function with points $(3, -4), (-3, 2), (1, 2)$.

► *Solution*: Let $f(x) = a(x-h)^2 + k$.

$(-3, 2)$ and $(1, 2)$ are symmetric because they have the same y-value. Therefore the axis of symmetry must be a vertical line equidistant from the two points. Its equation is $x = \frac{-3+1}{2} = -1$

Therefore $f(x) = a(x+1)^2 + k$.

Substituting $(1, 2)$ into the equation:

$$\begin{aligned} f(1) = a(1+1)^2 + k &= 2 \\ 4a + k &= 2 \\ k &= 2 - 4a \end{aligned}$$

Therefore $f(x) = a(x+1)^2 + 2 - 4a$.

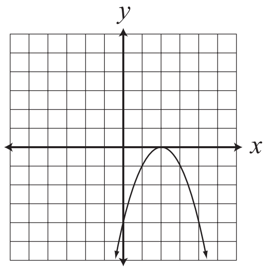
Using $(3, -4)$:

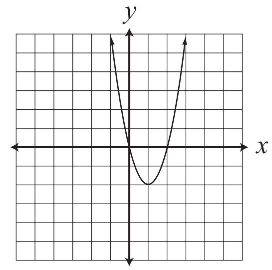
$$\begin{aligned} f(3) = a(3+1)^2 + 2 - 4a &= -4 \\ 16a + 2 - 4a &= -4 \\ 12a &= -6 \\ a &= -\frac{1}{2} \end{aligned}$$

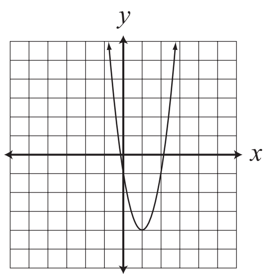
Therefore $k = 4$, and $f(x) = -\frac{1}{2}(x+1)^2 + 4$.

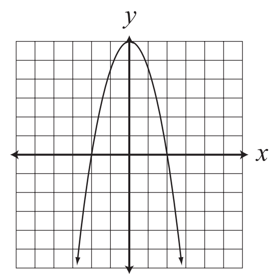
2.1 Exercise Set

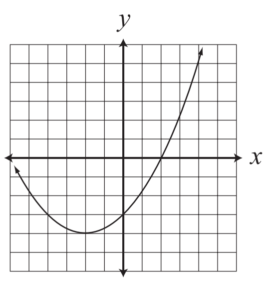
1. Determine the equation for the parabola.

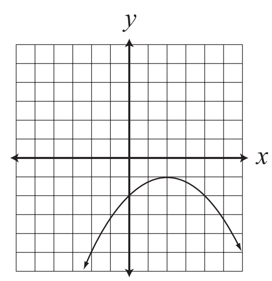
a)


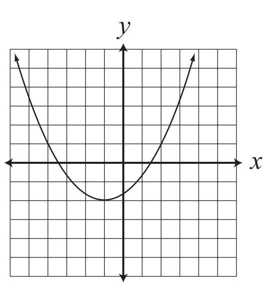
b)


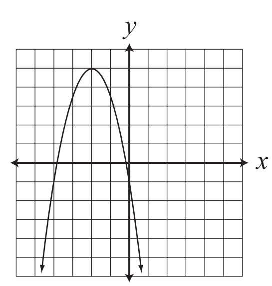
c)


d)


e)


f)


g)


h)


2. Find the equation of a quadratic function whose graph satisfies the given conditions.

a) vertex: $(2, 9)$ x-intercept: 5

b) vertex: $(-2, 12)$ x-intercept: -4

c) vertex: $(1, -4)$ x-intercept: -2

d) vertex: $(-4, 12)$ x-intercept: 4

e) vertex $(-3, -5)$ y-intercept: 1

f) vertex: $(2, 4)$ y-intercept: -3

g) vertex: $(1, 4)$ point: $(2, 3)$

h) vertex: $(-2, -4)$ point $(-3, -1)$

i) vertex: $(0, 2)$ point: $\left(\frac{1}{2}, \frac{3}{2}\right)$

j) vertex: $(-3, 0)$ point: $\left(-\frac{3}{2}, \frac{1}{2}\right)$

k) vertex: $(\sqrt{2}, 5)$ point: $(-3\sqrt{2}, -3)$

l) vertex: $(-\sqrt{3}, -6)$ point: $(3\sqrt{3}, 2)$

3. Find the equation of a quadratic function whose graph contains the given points.

a) $(2,0),(-4,0),(-1,-2)$

b) $(-3,0),(1,-8),(5,0)$

c) $(-1,0),(0,3),(3,0)$

d) $(-1,0),(0,5),(5,0)$

e) $(-1,0),\left(0,\frac{5}{2}\right),(5,0)$

f) $(-5,0),(-1,0),\left(0,\frac{5}{2}\right)$

g) $(-2,3),(0,3),(-4,-5)$

h) $(-3,-1),(1,-1),(-2,5)$

i) $(-2,1),(-6,1),(2,-7)$

j) $(-4,8),(-3,23),(-8,8)$

4. If a parabola has x-intercepts a and b, what is the x-coordinate of the vertex?

5. Find all values of k so that the vertex of the graph $y = x^2 + kx + 16$ lies on the x-axis.

6. Determine the quadratic function f, with zeros -2 and 6, and range $y \leq 4$.

7. Determine the quadratic function f with points $(-3, 5)$ and $(1, 5)$, and range $y \geq -2$.

8. If the graph of the quadratic function $f(x) = ax^2 + bx + c$ passes through the origin, what is the value of c?

9. Consider the quadratic equation $y = ax^2 + ax + b$. Find the vertex if $a = 4b$.

10. How far from the origin is the vertex of the parabola $y = -2(x - 3)^2 + 4$?

11. Find the distance between the vertices of the parabolas $y = \frac{1}{2}(x + 2)^2 + 6$ and $y = -2(x - 4)^2 - 2$.

2.2 General Form to Standard Form

The vertex of a quadratic function is not obvious from its general form, $y = f(x) = ax^2 + bx + c$. However, in the standard form of a parabola, $y = f(x) = a(x-h)^2 + k$, the vertex, (h, k), is easily apparent. Changing the general form of a parabola to the standard form of a parabola, simplifies the process of finding the vertex, and to graphing the parabola.

Perfect Square Trinomials

A trinomial that is the square of a binomial is called a perfect square trinomial.

For example: $x^2 + 6x + 9 = (x+3)(x+3) = (x+3)^2$

$x^2 - 4x + 4 = (x-2)(x-2) = (x-2)^2$

In a perfect square trinomial:
1. The last term must be positive and a perfect square.
2. The first term must be a perfect square.
3. The middle term is the square root of the first term multiplied by the square root of the last term, then doubled. It can be positive or negative.

To find the missing constant term that makes a trinomial a perfect square:
1. If the coefficient of the x^2 term is not 1, factor that value from both terms.
2. Take the constant of the linear term, divide the value by 2, then square it.
3. The squared value is the constant term.

Example 1 Find the constant that will make each expression a perfect square trinomial.

a) $x^2 - 10x + \_\_\_\_$

b) $x^2 + 2x + \_\_\_\_$

c) $2x^2 - 8x + \_\_\_\_$

d) $-2x^2 - 12x + \_\_\_\_$

e) $\frac{1}{2}x^2 - 10x + \_\_\_\_$

► *Solution*: **a)** $x^2 - 10x + \underline{25} = (x-5)(x-5) = (x-5)^2$

b) $x^2 + 2x + \underline{1} = (x+1)(x+1) = (x+1)^2$

c) $2x^2 - 8x + \_\_\_ = 2(x^2 - 4x + \_\_\_) = 2(x^2 - 4x + 4) = 2(x-2)(x-2) = 2(x-2)^2$

d) $-2x^2 - 12x + \_\_\_ = -2(x^2 + 6x + \_\_\_) = -2(x^2 + 6x + 9) = -2(x+3)(x+3) = -2(x+3)^2$

e) $\frac{1}{2}x^2 - 10x + \_\_\_ = \frac{1}{2}(x^2 - 20x + \_\_\_) = \frac{1}{2}(x^2 - 20x + 100) = \frac{1}{2}(x-10)^2$

Constants are: **a)** 25 **b)** 1 **c)** 8 **d)** -18 **e)** 50

Changing from General Form to Standard Form

1. Group the variable terms and constant terms separately.
2. Factor the leading coefficient 'a' from the variable terms.
3. Determine a constant value that makes the variable terms a perfect square trinomial. Add and subtract the the value to the variable terms. (It is necessary to add and subtract the same value in order to keep the balance of the equation.)
4. Regroup the variable terms into a perfect square trinomial. When the value that is removed from the variable terms is regrouped, it must be multiplied by the coefficient from step 2.
5. Factor the trinomial as a perfect square and a constant term.

Example 2 Find the vertex and x-intercepts of $f(x) = -2x^2 + 4x - 5$.

► *Solution*:

$$\begin{aligned} f(x) &= (-2x^2 + 4x + \_\_\_\ ) - 5 \\ &= -2(x^2 - 2x + \_\_\_\ ) - 5 \\ &= -2(x^2 - 2x + 1 - 1) - 5 \\ &= -2(x^2 - 2x + 1) + 2 - 5 \quad \leftarrow \textit{When regrouping, } -2 \times (-1) = 2 \\ &= -2(x-1)(x-1) - 3 \\ &= -2(x-1)^2 - 3 \end{aligned}$$

The vertex is $(1, -3)$

At the x-intercepts, $y = f(x) = 0 \ \rightarrow \ -2(x-1)^2 - 3 = 0$

$$(x-1)^2 = -\frac{3}{2}$$

A square value cannot equal a negative value, therefore there is no x-intercept.

Example 3 Find the vertex and x-intercepts of $f(x) = 2x^2 + 5x - 3$.

► *Solution*:

$$\begin{aligned} f(x) &= (2x^2 + 5x + \_\_\_\ ) - 3 \\ &= 2\left(x^2 + \frac{5}{2}x + \_\_\_\ \right) - 3 \\ &= 2\left(x^2 + \frac{5}{2}x + \frac{25}{16} - \frac{25}{16}\right) - 3 \\ &= 2\left(x^2 + \frac{5}{2}x + \frac{25}{16}\right) - \frac{25}{8} - 3 \quad \leftarrow \textit{When regrouping, } 2 \times \left(-\frac{25}{16}\right) = -\frac{25}{8} \\ &= 2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8} \end{aligned}$$

The vertex is $\left(-\frac{5}{4}, -\frac{49}{8}\right)$

At the x-intercepts, $y = f(x) = 0 \ \rightarrow \ 2\left(x + \frac{5}{4}\right)^2 - \frac{49}{8} = 0$

$$\begin{aligned} \left(x + \frac{5}{4}\right)^2 &= \frac{49}{16} \\ x + \frac{5}{4} &= \pm\frac{7}{4} \\ x &= -\frac{5}{4} \pm \frac{7}{4} = \frac{1}{2}, -3 \end{aligned}$$

The x-intercepts are $\frac{1}{2}, -3$.

2.2 Exercise Set

1. Find a number that makes the equation valid.

a) $(x + 1)^2 = x^2 + 2x +$ ____

b) $(x - 1)^2 = x^2 - 2x +$ ____

c) $(x + 2)^2 = x^2 + 4x +$ ____

d) $(x - 2)^2 = x^2 - 4x +$ ____

e) $(x + 3)^2 = x^2 + 6x +$ ____

f) $(x - 3)^2 = x^2 - 6x +$ ____

g) $(x + 4)^2 = x^2 + 8x +$ ____

h) $(x - 4)^2 = x^2 - 8x +$ ____

2. Find a number that makes the expression a perfect square of the form $(x + h)^2$.

a) $x^2 + 2x +$ ____

b) $x^2 - 2x +$ ____

c) $x^2 + 4x +$ ____

d) $x^2 - 4x +$ ____

e) $x^2 + 5x +$ ____

f) $x^2 - 7x +$ ____

g) $x^2 + bx +$ ____

h) $x^2 - bx +$ ____

3. Find a number to make each side equal in value.

a) $x^2 + 6x +$ ____ $= (x +$ ____ $)^2$

b) $x^2 - 10x +$ ____ $= (x -$ ____ $)^2$

c) $x^2 + 5x +$ ____ $= (x +$ ____ $)^2$

d) $x^2 - 7x +$ ____ $= (x -$ ____ $)^2$

e) $x^2 + \frac{2}{3}x +$ ____ $= (x +$ ____ $)^2$

f) $x^2 - \frac{3}{4}x +$ ____ $= (x -$ ____ $)^2$

g) $x^2 + bx +$ ____ $= (x +$ ____ $)^2$

h) $x^2 - \frac{b}{a}x +$ ____ $= (x -$ ____ $)^2$

4. Find the vertex.

a) $f(x) = x^2 + 4x + 3$

b) $g(x) = x^2 + 6x + 10$

c) $h(x) = x^2 - 8x + 15$

d) $j(x) = x^2 - 10x + 18$

e) $k(x) = x^2 + 3x - 8$

f) $l(x) = -x^2 - 3x + 5$

g) $m(x) = 3x^2 - 18x + 25$

h) $n(x) = -3x^2 + 5x - 3$

i) $p(x) = \frac{1}{2}x^2 - 3x + 4$

j) $q(x) = -\frac{2}{3}x^2 + 4x - 1$

k) $r(x) = 0.6x^2 + 2x - 3$

l) $s(x) = -\frac{3}{4}x^2 + \frac{2}{3}x + \frac{1}{2}$

5. Find the x and y-intercepts, if possible, for each quadratic function.

a) $f(x) = (x-3)^2 - 4$

b) $g(x) = (x-3)^2 + 4$

c) $h(x) = 4(x+1)^2 - 36$

d) $j(x) = -2(x+3)^2 + 11$

e) $k(x) = -\frac{1}{3}\left(x - \frac{1}{2}\right)^2$

f) $l(x) = x^2 + 5x$

g) $m(x) = -2x^2 - 4x - 6$

h) $n(x) = \frac{1}{2}x^2 - 3x + \frac{5}{2}$

i) $p(x) = -3x^2 - 5x$

j) $q(x) = -\frac{1}{2}x^2 - 4x - \frac{19}{3}$

k) $r(x) = -0.4x^2 + 2$

l) $s(x) = 6x^2 - \sqrt{2}\,x - 2$

6. Graph each quadratic function. Find the vertex, axis of symmetry, x-intercept(s), and y-intercept of the graph.

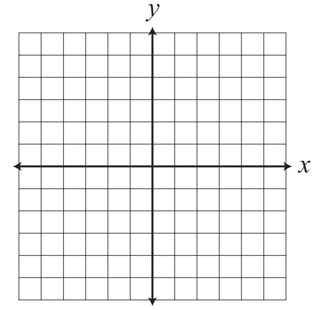
a) $f(x) = -x^2 - 4x$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



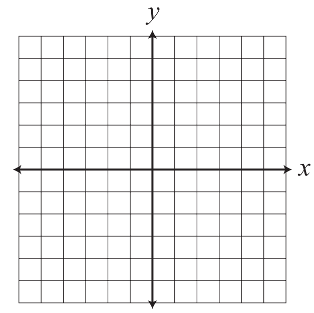
b) $g(x) = -x^2 + 4x - 3$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



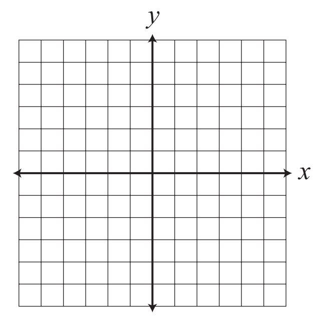
c) $h(x) = x^2 - 4x + 5$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



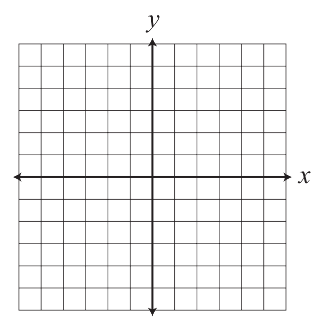
d) $i(x) = -2x^2 - 4x + 3$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



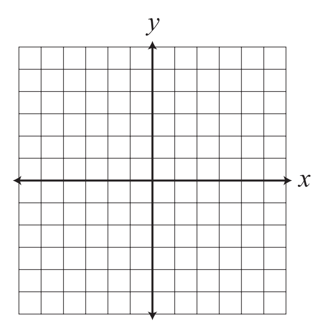
e) $j(x) = 3x - x^2$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



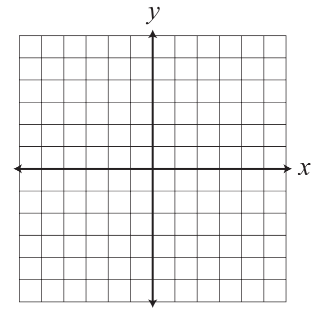
6. **f)** $k(x) = -2x^2 - 8x - 6$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



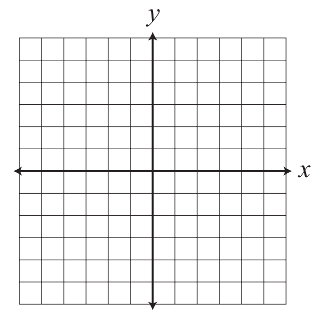
g) $l(x) = -3x^2 + 6x + 3$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



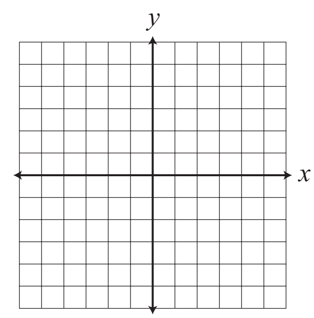
h) $m(x) = -\frac{1}{3}x^2 - 2x + 1$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



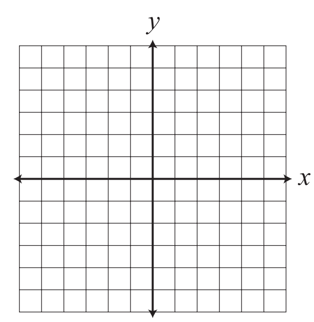
i) $n(x) = |x^2 - 4|$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



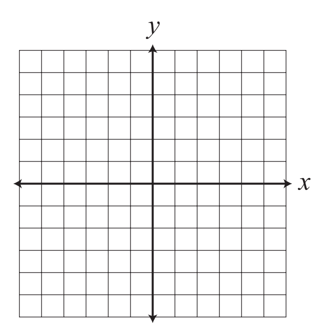
j) $p(x) = |-x^2 + 2x + 3|$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



7. Find the vertex of $f(x) = x^2 + kx + 4$.

8. Find the vertex of $g(x) = 2x^2 + kx + k^2$.

9. Find the vertex of $h(x) = 2x^2 + ax + b^2$.

10. Find the vertex of $i(x) = px^2 - 3x + p$.

11. Find the vertex of $j(x) = kx(8 - x)$.

12. Find a such that $f(x) = ax^2 + 4x - 4$ has a maximum value at $x = 6$.

13. Find b so that the function $f(x) = 2x^2 + bx - 3$ has a minimum value of -5.

14. Find c so that the function $f(x) = 0.1x^2 + 7x + c$ has a minimum value of -120.5.

2.3 Vertex of a Parabola

The vertex of a parabola in general form can be found using the vertex formula. The vertex formula is derived by completing the square of the general form of a parabola, $f(x) = ax^2 + bx + c$.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= (ax^2 + bx + \_\_\_\,) + c \\ &= a\left(x^2 + \frac{b}{a}x + \_\_\_\,\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + a\left(-\frac{b^2}{4a^2}\right) + c \\ &= a\left(x + \frac{b}{2a}\right)\left(x + \frac{b}{2a}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \end{aligned}$$

with $h = -\dfrac{b}{2a}$ and $k = c - \dfrac{b^2}{4a}$

The Vertex Formula

The graph of the function $f(x) = ax^2 + bx + c,\ a \neq 0$ is a parabola with vertex $\left(-\dfrac{b}{2a}, c - \dfrac{b^2}{4a}\right)$

Note: It is easier to find the y-coordinate by calculating the value of the x-coordinate, then substituting the value $-b/2a$, for x in the function.

Example 1 Find the vertex for $f(x) = -\frac{1}{2}x^2 + 4x - 3$.

► *Solution*: Using the vertex formula: $a = -\frac{1}{2}$, $b = 4$, and $c = -3$

$$x = -\frac{b}{2a} = -\frac{4}{2(-\frac{1}{2})} = 4$$

$$\begin{aligned} y &= c - \frac{b^2}{4a} \\ &= -3 - \frac{4^2}{4(-\frac{1}{2})} \\ &= 5 \end{aligned} \qquad \text{or} \qquad \begin{aligned} y &= -\frac{1}{2}x^2 + 4x - 3 \\ &= -\frac{1}{2}(4)^2 + 4(4) - 3 \\ &= 5 \end{aligned}$$

The vertex is $(4, 5)$.

Example 2 Graph the function $f(x)=\frac{1}{2}x^2-3x+\frac{5}{2}$.

► *Solution*: Using the vertex formula: $a=\frac{1}{2}$, $b=-3$, and $c=\frac{5}{2}$

$$x=-\frac{b}{2a}=-\frac{(-3)}{2(\frac{1}{2})}=3$$

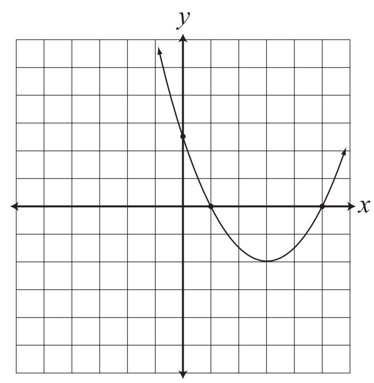
$$\begin{aligned} y &= c-\frac{b^2}{4a} \\ &= \frac{5}{2}-\frac{(-3)^2}{4(\frac{1}{2})} \\ &= -2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} y &= \frac{1}{2}x^2-3x+\frac{5}{2} \\ &= \frac{1}{2}(3)^2-3(3)+\frac{5}{2} \\ &= -2 \end{aligned}$$

The vertex is $(3,-2)$

y-intercept: $f(0)=\frac{1}{2}(0)^2-3(0)+\frac{5}{2}=\frac{5}{2}\rightarrow\left(0,\frac{5}{2}\right)$

x-intercept: $f(x)=\frac{1}{2}x^2-3x+\frac{5}{2}=0$

$$x^2-6x+5=0$$
$$(x-1)(x-5)=0$$
$$x=1,5\rightarrow(1,0),(5,0)$$



Example 3 Graph the function $f(x)=-2x^2+4x+3$.

► *Solution*: Using the vertex formula: $a=-2$, $b=4$, and $c=3$

$$x=-\frac{b}{2a}=-\frac{4}{2(-2)}=1$$

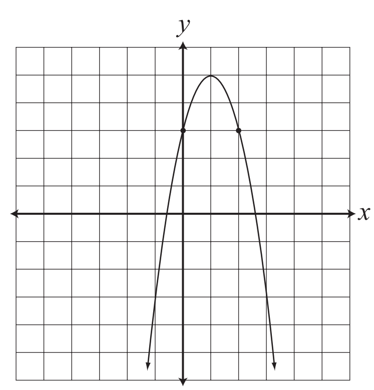
$$\begin{aligned} y &= c-\frac{b^2}{4a} \\ &= 3-\frac{4^2}{4(-2)} \\ &= 5 \end{aligned} \qquad \text{or} \qquad \begin{aligned} y &= -2x^2+4x+3 \\ &= -2(1)^2+4(1)+3 \\ &= 5 \end{aligned}$$

The vertex is $(1,5)$

y-intercept: $f(0)=-2(0)^2+4(0)+3=3\rightarrow(0,3)$

x-intercept: $f(x)=-2x^2+4x+3=0$

By quadratic formula: $x=\frac{2\pm\sqrt{10}}{2}\rightarrow\left(\frac{2\pm\sqrt{10}}{2},0\right)$



2.3 Exercise Set

1. Fill in the blanks to make a true statement.

a) A quadratic function in general form is defined by the equation ________________ .

b) The x-coordinate of the vertex of $y = ax^2 + bx + c$ is ________________ .

c) The y-coordinate of the vertex of $y = ax^2 + bx + c$ is ________________ .

d) The axis of symmetry of $y = ax^2 + bx + c$ is ________________ .

e) The ________________ is the lowest point on a parabola that opens up, or the highest point on a parabola that opens down.

f) The axis of symmetry of a parabola is a ________________ line through the vertex.

g) A parabola is the graph of a ________________ function in two variables.

h) If the vertex of the graph of $f(x) = ax^2 + bx + c$ is above the x-axis, and $a > 0$, then the graph of

$f(x) = 0$ has ________________ intercept(s).

i) If the vertex of the graph of $f(x) = ax^2 + bx + c$ is above the x-axis, and $a < 0$, then the graph of

$f(x) = 0$ has ________________ intercept(s).

j) If the graph of $f(x) = ax^2 + bx + c$ has $c - \frac{b^2}{4a} = 0$, then the graph has ________________ intercept(s).

2. Write a two-variable quadratic function, $f(x) = ax^2 + bx + c$, with the given values of coefficients a, b, c.

a) $a = 3, \quad b = -1, \quad c = 4$

b) $a = -\frac{1}{2}, \quad b = -3, \quad c = 2$

c) $a = -2, \quad b = 3, \quad c = 0$

d) $a = -\frac{2}{3}, \quad b = 0, \quad c = 3$

3. Find the vertex of each parabola.

a) $f(x) = x^2 - 4$

b) $g(x) = -x^2 + 3$

c) $h(x) = x^2 - 6x + 9$

d) $i(x) = x^2 + 6x - 3$

e) $j(x) = 3x^2 + 24x + 30$

f) $k(x) = \frac{1}{2}x^2 - 2x - 7$

g) $l(x) = -2x^2 + 12x - 17$

h) $m(x) = 2x^2 + 16x + 33$

i) $n(x) = 2x^2 - 5x - 3$

j) $p(x) = -\frac{1}{3}x^2 + 5x - 1$

4. Find the maximum or minimum value of each parabola.

a) $f(x) = 3x^2 - 7x + 8$

b) $f(x) = -\frac{1}{2}x^2 + 4x - 5$

c) $f(x) = -5x^2 - 10x + 3$

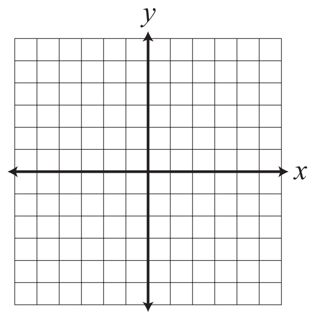
d) $f(x) = 4x^2 - x$

e) $f(x) = x^2 - 2x - 8$

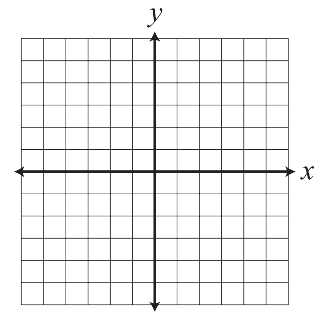
f) $f(x) = 1 - 3x^2$

5. Sketch the graph of each function f. Label the vertex plus four other points.

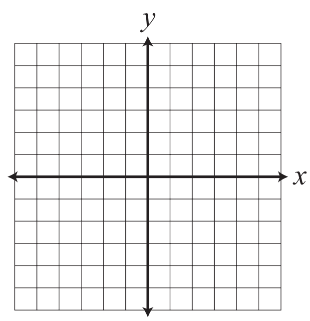
a) $f(x) = x^2 - 2x - 3$



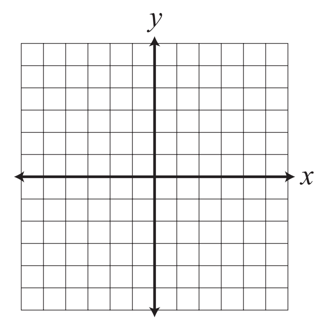
b) $f(x) = 2x^2 + 3x - 2$



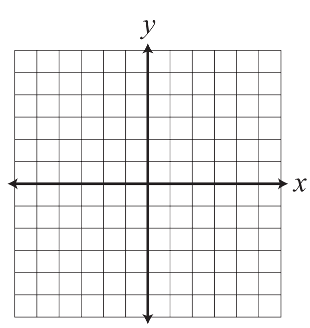
c) $f(x) = -3x^2 - 4x + 4$



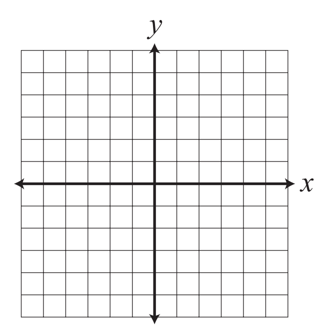
d) $f(x) = -4x^2 + 12x - 5$



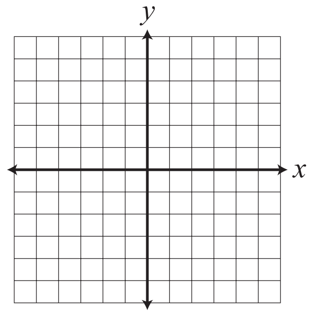
e) $f(x) = x^2 + x - 3.75$



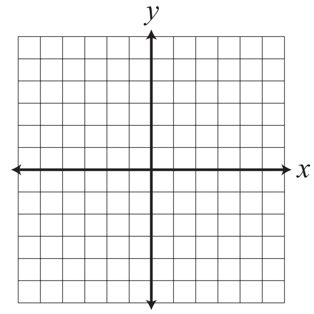
f) $f(x) = -\frac{1}{3}x^2 - 2x + 1$



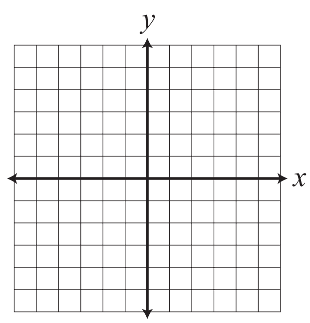
5. **g)** $f(x) = 3 + 5x - 2x^2$



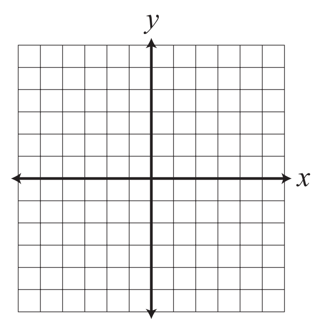
h) $f(x) = 3x^2 - 4x + 1$



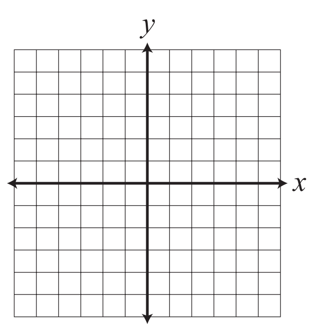
i) $f(x) = -4x^2 + 8x$



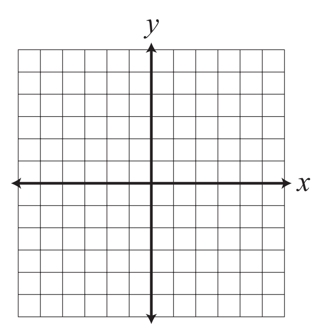
j) $f(x) = 2x^2 + 4x + 5$



k) $f(x) = 3x^2 + 3x - 5$



l) $f(x) = 2x^2 + 2x - 4$



6. Find the vertex of each quadratic function.

a) $f(x) = -bx^2 + cx - a$

b) $g(x) = cx^2 - ax + b$

7. Correct the error in the following statements.

a) The axis of symmetry of a parabola is 2 if the vertex is $(2, 4)$.

b) The vertex of $f(x) = x^2 - x - 12$ and $g(x) = 12 + x - x^2$ are equivalent since the axis of symmetry and x-intercepts are equal.

c) If the vertex is not on the x-axis, the parabola can have 0, 1, or 2 x-intercepts.

d) The axis of symmetry is the horizontal line through the vertex.

e) The graph of a parabola can have 0 or 1 y-intercepts.

f) The vertex of $y = 2(x + 3)^2 + 4$ is $(3, 4)$.

8. A baseball is projected upwards at an angle from ground level. Its height $s(x)$ is given by the function $s(x) = -\frac{1}{400}x^2 + x$, where x is the distance the ball travelled horizontally in feet.

a) What is the maximum height reached by the ball?

b) How far did the ball travel horizontally?

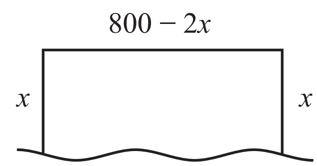
2.4 Applications of Quadratic Functions

There are several applications of maximum/minimum problems that can be modeled using quadratic functions.

Example 1 A rancher has 800 m of fencing to enclose a rectangular cattle pen along a river bank. There is no fencing needed along the river bank. Find the dimensions that would enclose the largest area.

► *Solution*: If $x =$ the width of the pen, the length of the pen is $800 - 2x$.

$$\begin{aligned}\text{Area} &= \text{length} \times \text{width} \\ &= x(800 - 2x) \\ &= 800x - 2x^2\end{aligned}$$



Method 1:
$$\begin{aligned}\text{Area} &= 800x - 2x^2 \\ &= -2(x^2 - 400x) \\ &= -2(x^2 - 400x + 200^2 - 200^2) \\ &= -2(x^2 - 400x + 200^2) - (-2)(-200)^2 \\ &= -2(x - 200)^2 + 80\,000\end{aligned}$$

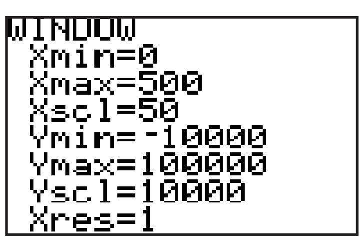
Method 2: Area $= 800x - 2x^2$

Using the vertex formula: $a = -2$, $b = 800$, $c = 0$

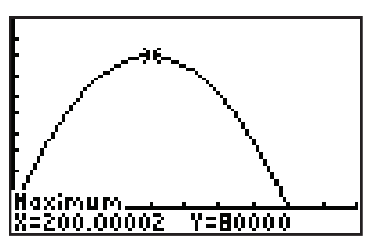
Vertex: $\left(\dfrac{-b}{2a}, c - \dfrac{b^2}{4a}\right) = \left(\dfrac{-800}{2(-2)}, 0 - \dfrac{800^2}{4(-2)}\right) = (200, 80\,000)$

Method 3: Using a graphing calculator, graph $y = 800x - 2x^2$

Set window:



Calculate maximum:



The maximum area of the pen = 200 m × 400 m = 80 000 m².

Example 2 A small manufacturer of windows has a daily production cost of $C = 300 - 4x + 0.2x^2$, where C is the total cost in dollars and x is the number of units made. How many windows should be produced every day to minimize costs?

► *Solution*: Method 1:

$$\begin{aligned} C &= 0.2x^2 - 4x + 300 \\ &= 0.2(x^2 - 20x + \_\_\_) + 300 \\ &= 0.2(x^2 - 20x + 100 - 100) + 300 \\ &= 0.2(x^2 - 20x + 100) - 20 + 300 \\ &= 0.2(x - 10)^2 + 280 \end{aligned}$$

Vertex is a minimum at $(10, 280)$

Method 2: $C = 300 - 4x + 0.2x^2$

Using the vertex formula: $a = 0.2$, $b = -4$, $c = 300$

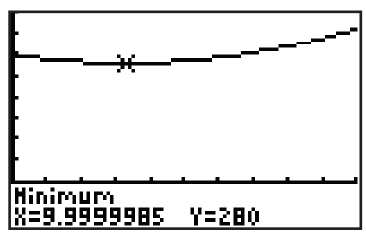
Vertex: $\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right) = \left(\frac{-(-4)}{2(0.2)}, 300 - \frac{(-4)^2}{4(0.2)}\right) = (10, 280)$

Method 3: Using a graphing calculator, graph $y = 300 - 4x + 0.2x^2$

Set window:

```
WINDOW
 Xmin=0
 Xmax=30
 Xscl=3
 Ymin=-100
 Ymax=400
 Yscl=50
 Xres=1
```

Calculate maximum:



The manufacturer should produce 10 windows per day at a production cost of $280.00.

Example 3 A 400 room hotel is three-quarters full when the room rate is an average of \$80.00 per night. A survey shows that each \$5.00 increase in cost will result in 10 fewer customers. Find the nightly rate and number of rooms occupied that will maximize income.

► *Solution*: $\frac{3}{4}$ of $400 = 300$ rooms occupied.

Income = number of rooms occupied × cost per night.

Let $I(x)$ = income from hotel.

Let $x =$ the number of \$5 increases in price.

The average cost of a hotel room is $(80 + 5x)$.

The number of rooms rented is $(300 - 10x)$.

Therefore $I(x) = (300 - 10x)(80 + 5x) = 24\,000 + 700x - 50x^2$.

<u>Method 1</u>: Income $= 24\,000 + 700x - 50x^2$

$$\begin{aligned} &= -50(x^2 - 14x + \underline{\quad}) + 24\,000 \\ &= -50(x^2 - 14x + 49 - 49) + 24\,000 \\ &= -50(x^2 - 14x + 49) + 50 \cdot 49 + 24\,000 \\ &= -50(x - 7)^2 + 26\,450 \end{aligned}$$

Vertex is a maximum at $(7, 26\,450)$

<u>Method 2</u>: Income $= 24\,000 + 700x - 50x^2$

Using the vertex formula: $a = -50$, $b = 700$, $c = 24\,000$

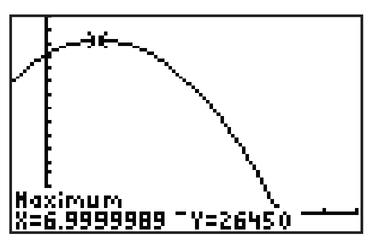
Vertex: $\left(\frac{-b}{2a}, c - \frac{b^2}{4a}\right) = \left(\frac{-700}{2(-50)}, 24\,000 - \frac{700^2}{4(-50)}\right) = (7, 26\,450)$

<u>Method 3</u>: Using a graphing calculator, graph $y = 24\,000 + 700x - 50x^2$

Set window:

```
WINDOW
 Xmin=-4
 Xmax=40
 Xscl=4
 Ymin=-2000
 Ymax=30000
 Yscl=2000
 Xres=1
```

Calculate maximum:



The nightly rate should average $\$80 + 5(7) = \$115$

The number of rooms occupied would be $300 - 10(7) = 230$

The maximum income is $230 \times 115 = \$26\,450$

Example 4 Murray stands on the top of a building and fires a gun upwards. The bullet travels according to the equation $h = -16t^2 + 384t + 50$, where h is the height of the bullet off the ground in metres at t seconds after it was fired.

a) How far is Murray above the ground when he shoots the gun?

b) How high does the bullet travel vertically relative to the ground?

c) How long does it take for the bullet to reach its greatest height?

d) After how many seconds does the bullet hit the ground?

► *Solution*: **a)** when $t = 0$, $h = 50$ m.

b)
$$\begin{aligned} h &= -16t^2 + 384t + 50 \\ &= -16(t^2 - 24t + \_\_\_) + 50 \\ &= -16(t^2 - 24t + 144 - 144) + 50 \\ &= -16(t^2 - 24t + 144) + 2304 + 50 \\ &= -16(t - 12)^2 + 2354 \end{aligned}$$

Vertex is a maximum at $(12, 2345)$, therefore the bullet travels 2354 m high.

c) $t - 12 = 0 \rightarrow t = 12$ seconds.

d) The bullet reaches the ground when $h = 0$.

$$\begin{aligned} -16(t - 12)^2 + 2354 &= 0 \\ (t - 12)^2 &= \frac{2354}{16} \\ t - 12 &= \pm\sqrt{\frac{2354}{16}} \\ t &= 12 \pm \sqrt{\frac{2354}{16}} \\ t &= 24.13 \text{ seconds or } -0.13 \text{ seconds.} \end{aligned}$$

Reject $t = -0.13$ seconds, therefore it takes the bullet 24.13 seconds to hit the ground.

2.4 Exercise Set

1. The Acme automobile company has found that the revenue from sales of cars is a function of the unit price p that it charges. If the revenue, R, is $R = -\frac{1}{2}p^2 + 2000p$, what unit price, p, should be charged to maximize revenue? What is the maximum revenue?

2. A cattle ranch with 6000 metres of fencing wants to enclose a rectangular feedlot that borders on a river. If the cattle will not go in the river, what is the largest area that can be enclosed?

3. Find the rectangle of maximum area that can be constructed with a perimeter of 36 cm.

4. What is the minimum product of two numbers that differ by 8? What are the numbers?

5. A cannon shell is fired from a cliff 100 m above the water. The height, h, of the cannon above the water is given by $h = -0.005x^2 + x + 100$, where x is the horizontal distance of the cannon from the base of the cliff in metres.

a) How far from the base of the cliff is the height of the cannon shell at a maximum height?

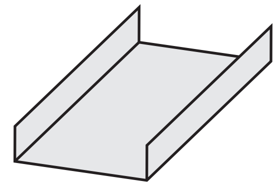
b) Find the maximum height of the cannon shell.

c) How far from the base of the cliff will the cannon shell land in the water?

6. The school play charges \$10 for admission, and on average 80 people attend each show. For each \$1 increase, attendance drops by 5 people. What price should the school charge to maximize revenue?

7. A company can sell x stereos at a price of $\$(500 - x)$. How many stereos must be sold to maximize income? Find the maximum income.

8. A 20 cm wide sheet of metal is bent into a rectangular trough. How high must the trough be to maximize the trough's volume?



9. The sum of two integers is 10, and the sum of their squares is a minimum. Find the integers.

10. A parabolic arch has a span of 50 m and a maximum height of 10 m. What is the height of the arch at a point 10 m from the centre?

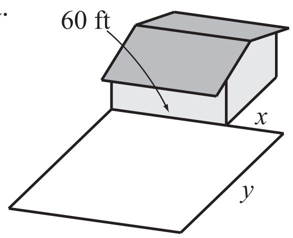
11. A rancher has 1200 m of fencing to enclose two adjacent rectangular corrals. What dimensions will produce a maximum enclosed area if the common sides are of equal length?

12. Semiahmoo Fish and Game Club charges its members \$200 per year. For each increase over 60 members, the membership cost is decreased by \$2. What number of members would produce the maximum revenue for the club?

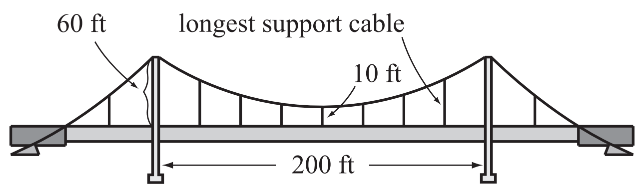
13. When priced at $10, one type of software has annual sales of 300 units. For each dollar the software is increased in price, the store expects to lose the sale of 10 units of software. Find the price that will maximize the total revenue.

14. Ship A is 50 nautical miles west of ship B. Ship A is heading east at 10 knots and ship B is heading south at 5 knots. Find the minimum distance between the ships, and at what time it occurred.

15. An equestrian club wants to construct a corral next to the barn, using the side of the barn. They have 300 ft of fencing. Find the dimensions that give the maximum area.



16. The cable supporting a suspension bridge is parabolic in shape. The support cables are vertical cables 25 ft apart. If the shortest support cable is 10 ft long, how long is the longest support cable?



17. A Norman window is a rectangle with a semi-circle on top. If the perimeter of the window is 24 ft, what dimensions will maximize the area of the window?

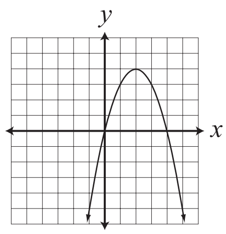
18. A piece of string 36 cm in length is cut into two pieces. One piece forms a square and the other piece a circle. How should the string be cut so the sum of the areas are a minimum?

2.5 Chapter Review

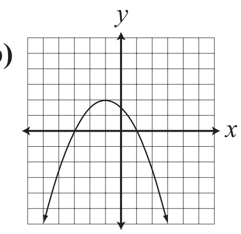
Section 2.1

1. Determine the equation of the parabola in the form $f(x) = a(x - h)^2 + k$.

a)



b)



2. Find the equation of a quadratic function whose graph satisfies the given conditions.

a) vertex: $(3, -4)$, x-intercept: 2

b) vertex: $(-2, 1)$, x-intercept: -3

c) vertex: $(-1, -4)$, point: $(2, -1)$

d) $(-2, 0), (4, 0), (0, 3)$

Section 2.2

3. Graph each quadratic function. Find the vertex, axis of symmetry, x-intercept(s), and y-intercept of the graph.

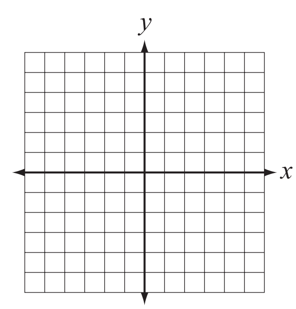
a) $f(x) = -\frac{1}{3}x^2 + x$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



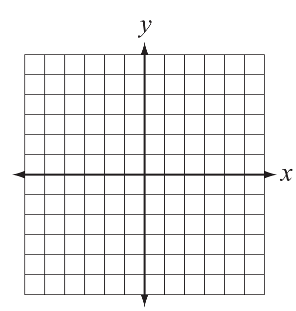
b) $g(x) = \frac{1}{2}x^2 - x - 4$

vertex __________

axis of symmetry __________

x-intercept(s) __________

y-intercept __________



4. Find the vertex, x-intercept(s) (if possible), and y-intercepts for each quadratic function.

a) $f(x) = -2x^2 - 6x - 4$

b) $g(x) = \frac{1}{2}x^2 - \frac{5}{2}x + \frac{25}{8}$

Section 2.3

5. Find the vertex of each quadratic function.

a) $f(x) = 2x^2 - 12x + 7$

b) $g(x) = 3x^2 + 6x + 2$

c) $h(x) = -2x^2 + 12x - 20$

d) $i(x) = -2 + 2x - \frac{1}{2}x^2$

6. If $(8, 0)$ is on $y = bx + x^2$, what is the least value of the function?

7. If $(-6, 0)$ is on $y = bx - x^2$, what is the greatest value of the function?

Section 2.4

8. Find the rectangle of maximum area that can be constructed with a perimeter of 44 cm.

9. A bus touring company charges \$10 per passenger, and carries an average of 300 passengers per day. The company estimates it will lose 15 passengers for each increase of \$1 in the fare. What is the most profitable fare to charge?