Quadratic Transformations - Solutions

2.1 Finding the Equation of a Parabola

1. **a)** $f(x) = -(x-2)^2$ **b)** $f(x) = 2(x-1)^2 - 2$ **c)** $f(x) = 3(x-1)^2 - 4$ **d)** $f(x) = -\frac{3}{2}x^2 + 6$

e) $f(x) = \frac{1}{4}(x+2)^2 - 4$ **f)** $f(x) = -\frac{1}{4}(x-2)^2 - 1$ **g)** $f(x) = \frac{1}{3}(x+1)^2 - 2$ **h)** $f(x) = -\frac{3}{2}(x+2)^2 + 5$

2. **a)** $f(x) = -(x-2)^2 + 9$ **b)** $f(x) = -3(x+2)^2 + 12$ **c)** $f(x) = \frac{4}{9}(x-1)^2 - 4$ **d)** $f(x) = -\frac{3}{16}(x+4)^2 + 12$

e) $f(x) = \frac{2}{3}(x+3)^2 - 5$ **f)** $f(x) = -\frac{7}{4}(x-2)^2 + 4$ **g)** $f(x) = -(x-1)^2 + 4$ **h)** $f(x) = 3(x+2)^2 - 4$

i) $f(x) = -2x^2 + 2$ **j)** $f(x) = \frac{2}{9}(x+3)^2$ **k)** $f(x) = -\frac{1}{4}(x-\sqrt{2})^2 + 5$ **l)** $f(x) = \frac{1}{6}(x+\sqrt{3})^2 - 6$

3. **a)** $f(x) = \frac{2}{9}(x+1)^2 - 2$ **b)** $f(x) = \frac{1}{2}(x-1)^2 - 8$ **c)** $f(x) = -(x-1)^2 + 4$ **d)** $f(x) = -(x-2)^2 + 9$

e) $f(x) = -\frac{1}{2}(x-2)^2 + \frac{9}{2}$ **f)** $f(x) = \frac{1}{2}(x+3)^2 - 2$ **g)** $f(x) = -(x+1)^2 + 4$ **h)** $f(x) = -2(x+1)^2 + 7$

i) $f(x) = -\frac{1}{4}(x+4)^2 + 2$ **j)** $f(x) = 3(x+6)^2 - 4$

4. The x-intercept is the midpoint of a and b, therefore $x = \frac{a+b}{2}$.

5. The x-axis has $y = 0$; $x^2 + kx + 16 = 0$ must be a perfect square of $(x+4)^2$ or $(x-4)^2$, therefore $k = -8, 8$

6. If the zeros are -2 and 6, then the vertex is $\left(\frac{-2+6}{2}, 4\right) = (2, 4)$.

$f(x) = a(x-2)^2 + 4 \rightarrow f(6) = a(6-2)^2 + 4 = 0 \rightarrow a = -\frac{1}{4}$; $f(x) = -\frac{1}{4}(x-2)^2 + 4$

7. Points $(-3, 5)$ and $(1, 5)$ are horizontal, therefore the vertex is $\left(\frac{-3+1}{2}, -2\right) = (-1, -2)$.

$f(x) = a(x+1)^2 - 2 \rightarrow f(1) = a(1+1)^2 - 2 = 5 \rightarrow a = \frac{7}{4}$; $f(x) = \frac{7}{4}(x+1)^2 - 2$

8. To pass through the origin, c must be zero since $f(x)$ and x must be zero.

9. $y = ax^2 + ax + \frac{a}{4} = a\left(x^2 + x + \frac{1}{4}\right) = a\left(x + \frac{1}{2}\right)^2$, therefore the vertex is $\left(-\frac{1}{2}, 0\right)$

10. The vertex is $(3, 4)$. By Pythagoras $d^2 = 3^2 + 4^2 \rightarrow d = 5$.

11. The vertices are $(-2, 6)$ and $(4, -2)$. By Pythagoras $d^2 = (-2-4)^2 + (6+2)^2 \rightarrow d = 10$.

2.2 General Form to Standard Form

1. **a)** 1 **b)** 1 **c)** 4 **d)** 4 **e)** 9 **f)** 9 **g)** 16 **h)** 16

2. **a)** 1 **b)** 1 **c)** 4 **d)** 4 **e)** $\frac{25}{4}$ **f)** $\frac{49}{4}$ **g)** $\frac{b^2}{4}$ **h)** $\frac{b^2}{4}$

3. **a)** 9, 3 **b)** 25, 5 **c)** $\frac{25}{4}, \frac{5}{2}$ **d)** $\frac{49}{4}, \frac{7}{2}$ **e)** $\frac{1}{9}, \frac{1}{3}$ **f)** $\frac{9}{64}, \frac{3}{8}$ **g)** $\frac{b^2}{4}, \frac{b}{2}$ **h)** $\frac{b^2}{4a^2}, \frac{b}{2a}$

4. **a)** $(-2, -1)$ **b)** $(-3, 1)$ **c)** $(4, -1)$ **d)** $(5, -7)$ **e)** $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ **f)** $\left(-\frac{3}{2}, \frac{29}{4}\right)$ **g)** $(3, -2)$ **h)** $\left(\frac{5}{6}, -\frac{11}{12}\right)$

i) $\left(3, -\frac{1}{2}\right)$ **j)** $(3, 5)$ **k)** $\left(-\frac{5}{3}, -\frac{14}{3}\right)$ **l)** $\left(\frac{4}{9}, \frac{35}{54}\right)$

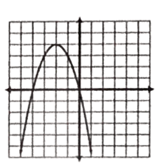
5. **a)** $(1,0), (5,0), (0,5)$ **b)** $(0, 13)$ **c)** $(2,0), (-4,0), (0,-32)$ **d)** $\left(\frac{-6 \pm \sqrt{22}}{2}, 0\right), (0, -7)$ **e)** $\left(\frac{1}{2}, 0\right), \left(0, -\frac{1}{12}\right)$

f) $(-5,0), (0,0)$ **g)** $(0, -6)$ **h)** $(1,0), (5,0), \left(0, \frac{5}{2}\right)$ **i)** $\left(-\frac{5}{3}, 0\right), (0,0)$ **j)** $\left(\frac{-12 \pm \sqrt{30}}{3}, 0\right), \left(0, -\frac{19}{3}\right)$

k) $(-\sqrt{5}, 0), (\sqrt{5}, 0), (0, 2)$ **l)** $\left(-\frac{\sqrt{2}}{3}, 0\right), \left(\frac{\sqrt{2}}{2}, 0\right), (0, -2)$

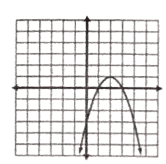
6. **a)**

vertex	$(-2, 4)$
axis of symmetry	$x = -2$
x-intercept(s)	$(-4, 0), (0, 0)$
y-intercept	$(0, 0)$



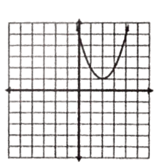
b)

vertex	$(2, 1)$
axis of symmetry	$x = 2$
x-intercept(s)	$(1, 0), (3, 0)$
y-intercept	$(0, -3)$



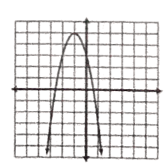
c)

vertex	$(2, 1)$
axis of symmetry	$x = 2$
x-intercept(s)	none
y-intercept	$(0, 5)$

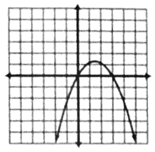


d)

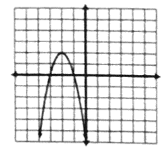
vertex	$(-1, 5)$
axis of symmetry	$x = -1$
x-intercept(s)	$\left(\frac{-2 \pm \sqrt{10}}{2}, 0\right)$
y-intercept	$(0, 3)$



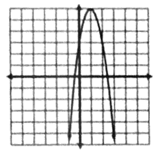
6. **e)** vertex $\left(\frac{3}{2}, \frac{9}{4}\right)$
axis of symmetry $x = \frac{3}{2}$
x-intercept(s) $(0,0), (3,0)$
y-intercept $(0,0)$



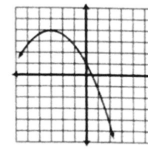
f) vertex $(-2, 2)$
axis of symmetry $x = -2$
x-intercept(s) $(-3, 0), (-1, 0)$
y-intercept $(0, -6)$



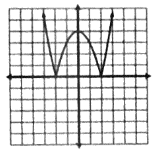
g) vertex $(1, 6)$
axis of symmetry $x = 1$
x-intercept(s) $(1 \pm \sqrt{2}, 0)$
y-intercept $(0, 3)$



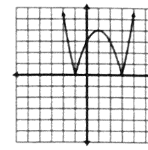
h) vertex $(-3, 4)$
axis of symmetry $x = -3$
x-intercept(s) $(-3 \pm 2\sqrt{3}, 0)$
y-intercept $(0, 1)$



i) vertex $(0, 4)$
axis of symmetry $x = 0$
x-intercept(s) $(-2, 0), (2, 0)$
y-intercept $(0, 4)$



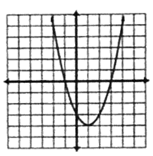
j) vertex $(1, 4)$
axis of symmetry $x = 1$
x-intercept(s) $(-1, 0), (3, 0)$
y-intercept $(0, 3)$



7. $\left(-\frac{k}{2}, 4 - \frac{k^2}{4}\right)$
8. $\left(-\frac{k}{4}, \frac{7k^2}{8}\right)$
9. $\left(-\frac{a}{4}, b^2 - \frac{a^2}{8}\right)$
10. $\left(\frac{3}{2p}, \frac{4p^2 - 9}{4p}\right)$
11. $(4, 16k)$
12. $a = -\frac{1}{3}$
13. $b = 4, -4$
14. $c = 2$

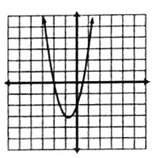
2.3 Vertex of a Parabola

1. **a)** $f(x) = ax^2 + bx + c,\ a \neq 0$ **b)** $-\frac{b}{2a}$ **c)** $c - \frac{b^2}{4a}$ **d)** $x = -\frac{b}{2a}$ **e)** vertex **f)** vertical **g)** quadratic **h)** 0 **i)** 2 **j)** 1
2. **a)** $f(x) = 3x^2 - x + 4$ **b)** $f(x) = -\frac{1}{2}x^2 - 3x + 2$ **c)** $f(x) = -2x^2 + 3x$ **d)** $f(x) = -\frac{2}{3}x^2 + 3$
3. **a)** $(0, -4)$ **b)** $(0, 3)$ **c)** $(3, 0)$ **d)** $(-3, -12)$ **e)** $(-4, -18)$ **f)** $(2, -9)$ **g)** $(3, 1)$ **h)** $(-4, 1)$ **i)** $\left(\frac{5}{4}, -\frac{49}{8}\right)$ **j)** $\left(\frac{15}{2}, \frac{71}{4}\right)$
4. **a)** min: $\frac{47}{12}$ **b)** max: 3 **c)** max: 8 **d)** min: $-\frac{1}{16}$ **e)** min: -9 **f)** max: 1
5. **a)**



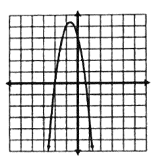
Vertex: $(1, -4)$

b)



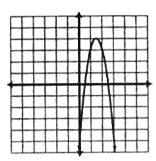
Vertex: $\left(-\frac{3}{4}, -\frac{25}{8}\right)$

c)



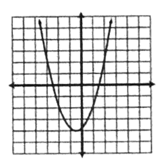
Vertex: $\left(-\frac{2}{3}, \frac{16}{3}\right)$

d)



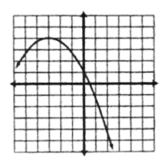
Vertex: $\left(\frac{3}{2}, 4\right)$

e)



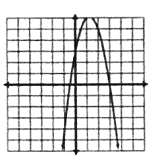
Vertex: $\left(-\frac{1}{2}, -4\right)$

f)



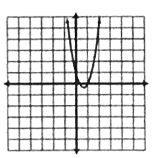
Vertex: $(-3, 4)$

g)



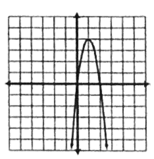
Vertex: $\left(\frac{5}{4}, \frac{49}{8}\right)$

h)



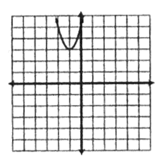
Vertex: $\left(\frac{2}{3}, -\frac{1}{3}\right)$

i)



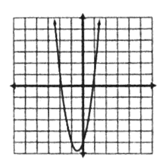
Vertex: $(1, 4)$

j)



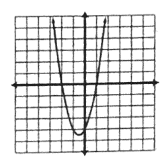
Vertex: $(-1, 3)$

k)



Vertex: $\left(-\frac{1}{2}, -\frac{23}{4}\right)$

l)



Vertex: $\left(-\frac{1}{2}, -\frac{9}{2}\right)$

6. **a)** $\left(\frac{c}{2b}, -a + \frac{c^2}{4b}\right)$ **b)** $\left(\frac{a}{2c}, b - \frac{a^2}{4c}\right)$
7. **a)** $x = 2$ **b)** the y-values are negatives of each other **c)** 0 or 2 x-intercepts **d)** vertical **e)** always 1 **f)** $(-3, 4)$
8. **a)** 100 ft. **b)** 400 ft.

2.4 Applications of Quadratic Functions

1. $R=-\frac{1}{2}p^2+2000p \to R=-\frac{1}{2}(p^2-4000p+\underline{\quad}) \to R-2\,000\,000=-\frac{1}{2}(p^2-4000p+4\,000\,000) \to$
$R=-\frac{1}{2}(p-2000)^2+2\,000\,000$; \$2000 should be charged, with a maximum revenue of \$2 000 000.00

2. $A=l\cdot w \to A=x(6000-2x) \to A=-2x^2+6000x \to A=-2(x^2-3000x+\underline{\quad}) \to$
$A-4\,500\,000=-2(x^2-3000x+2\,250\,000) \to A=-2(x-1500)^2+4\,500\,000$
The largest area enclosed is 4 500 000 m² = 4.5 km²

3. $A=x(18-x)=-(x^2-18x+81)+81=-(x-9^2)+81$; The rectangle is 9 cm × 9 cm, with area 81 cm²

4. $M=x(x+8)=(x^2+8x+16)-16=(x+4)^2-16$; $x=-4$, therefore the other number is $-4+8=4$.
Minimum product is $-4\times 4=-16$

5. $h=-0.005x^2+x+100 \to h-100=-0.005(x^2-200x+\underline{\quad}) \to$
$h-100-50=-0.005(x^2-200x+10\,000) \to h=-0.005(x-100)^2+150$
a) 100 m **b)** 150 m **c)** $h=-0.005(x-100)^2+150=0 \to x=100+\sqrt{30\,000}=273.21$ m

6. $P=(10+x)(80-5x)=-5x^2+30x+800$; $x=\frac{-b}{2a}=\frac{-30}{-10}=3$; The school should charge \$13.

7. $M=x(500-x)=-x^2+500x$; $x=\frac{-b}{2a}=\frac{-500}{2(-1)}=250$; The company must sell 250 stereos, for a max income of \$62 500.

8. $M=x(20-2x)=-2(x^2-10x+25)+50=-2(x-5)^2+50$; The trough must be 5 cm high.

9. Let x be one integer, then the other integer is $10-x$.
$S=x^2+(10-x^2) \to S=x^2+100-20x+x^2 \to S-100=2(x^2-10x+\underline{\quad}) \to$
$S-100+50=2(x^2-10x+25) \to S=2(x-5)^2+50$; Therefore $x=5$ and the other integer also is 5.

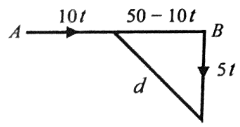
10. Let the top of the parabola be $(0,0)$; $y=a(x-0)^2+0 \to y=ax^2 \to -10=a(25)^2 \to$
$a=-\frac{10}{625}=-\frac{2}{125} \to y=-\frac{2}{125}x^2$; 10 metres from centre $y=-\frac{2}{125}(10)^2=-1.6$; The height is $10-1.6=8.4$ m.

11. $A=x\left(600-\frac{3}{2}x\right) \to A=-\frac{3}{2}x^2+600x \to A=-\frac{3}{2}(x^2-400x+\underline{\quad}) \to$
$A-60\,000=-\frac{3}{2}(x^2-400x+40\,000) \to A=-\frac{3}{2}(x-200)^2+60\,000$
$x=200$, length $=600-\frac{3}{2}(200)=300$; Therefore the maximum outside dimension is 200 m × 300 m.

12. Let x be the increase in members, and R be the revenue. The total number of members is $(60+x)$.
The fees would decrease by $2x$ to $(200-2x)$.
$R=(60+x)(200-2x) \to R=12\,000+80x-2x^2 \to R-12\,000=-2(x^2-40x+\underline{\quad}) \to$
$R-12\,000-800=-2(x^2-40x+400) \to R=-2(x-20)^2+12\,800$; 80 members would produce a max revenue of \$12 800.

13. $M=(10+x)(300-10x) \to M=-10x^2+200x+3000$; $x=\frac{-b}{2a}=\frac{-200}{-20}=10$; \$20 will maximize the total revenue.

14.

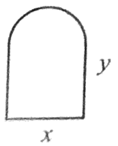


$d^2=(50-10t)^2+(5t)^2 \to d^2=2500-1000t+100t^2+25t^2 \to$
$d^2-2500=125(t^2-8t+\underline{\quad}) \to d^2-2500+2000=125(t^2-8t+16) \to$
$d^2=125(t-4)^2+500$
In 4 hours, the closest the two ships will be is $\sqrt{500}=22.36$ nautical miles apart.

15. $2y+2x+60=300 \to y=120-x$; $A=y(x+60) \to A=(120-x)(x+60) \to A=-x^2+60x+7200$;
$x=\frac{-b}{2a}=\frac{-60}{2(-1)}=30$; $y=120-x \to y=120-30=90$; The maximum area is 90 ft × 90 ft.

16. $y=ax^2 \to 50=a(100)^2 \to a=\frac{1}{200}$; $y=\frac{1}{200}x^2 \to y=\frac{1}{200}(75)^2=28.125+10=38.125$ ft long.

17.



$P=x+2y+\frac{\pi}{2}x=24$; $A=\frac{\pi}{8}x^2+xy=\frac{\pi}{8}x^2+x\left(12-\frac{x}{2}-\frac{\pi}{4}x\right)=-\left(\frac{\pi+4}{8}\right)x^2+12x$;
$x=\frac{-b}{2a}=\frac{-12}{-2\left(\frac{\pi+4}{8}\right)}=6.72$; $y=3.36$
6.72 ft × 3.36 ft plus the semi-circle will maximize the dimensions.

18. Let the circumference be x: $2\pi r = x \rightarrow r = \frac{x}{2\pi}$; $A = \pi r^2 = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$

Let the perimeter be $36 - x$: $A = \left(9 - \frac{1}{4}x\right)^2 = 81 - \frac{9}{2}x + \frac{1}{16}x^2$

Sum $= \left(\frac{1}{4}\pi + \frac{1}{16}\right)x^2 - \frac{9}{2}x + 81$; $x = \frac{-b}{2a} = \frac{\frac{9}{2}}{2\left(\frac{1}{4\pi} + \frac{1}{16}\right)} = 15.84$ cm

Therefore the circle piece should be 15.84 cm, and the square piece should be 20.16 cm to minimize the areas.

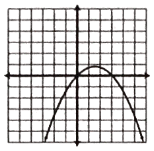
2.5 Chapter Review

1. **a)** $f(x) = -(x-2)^2 + 4$ **b)** $f(x) = -\frac{1}{2}(x+1)^2 + 2$

2. **a)** $y = 4(x-3)^2 - 4$ **b)** $y = -(x+2)^2 + 1$ **c)** $y = \frac{1}{3}(x+1)^2 - 4$ **d)** $y = -\frac{3}{8}(x-1)^2 + \frac{27}{8}$

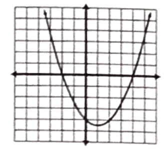
3. **a)**

vertex	$\left(\frac{3}{2}, \frac{3}{4}\right)$
axis of symmetry	$x = \frac{3}{2}$
x-intercept(s)	$(0,0), (3,0)$
y-intercept	$(0,0)$



b)

vertex	$\left(1, -\frac{9}{2}\right)$
axis of symmetry	$x = 1$
x-intercept(s)	$(-2,0), (4,0)$
y-intercept	$(0,-4)$



4. **a)**

vertex	$\left(-\frac{3}{2}, \frac{1}{2}\right)$
x-intercept(s)	$(-2,0), (-1,0)$
y-intercept	$(0,-4)$

b)

vertex	$\left(\frac{5}{2}, 0\right)$
x-intercept(s)	$\left(\frac{5}{2}, 0\right)$
y-intercept	$\left(0, \frac{25}{8}\right)$

5. **a)** $(3,-11)$ **b)** $(-1,-1)$ **c)** $(3,-2)$ **d)** $(2,0)$

6. -16

7. 9

8. $P = 2l + 2w = 44 \rightarrow l + w = 22 \rightarrow w = 22 - l$

$A = l \cdot w \rightarrow A = l(22-l) \rightarrow A = 22l - l^2 \rightarrow A = -(l^2 - 22l + 121) + 121 \rightarrow A = -(l-11)^2 + 121 \rightarrow$

$w = 22 - l = 22 - 11 = 11$

The rectangle of maximum area is 11 cm × 11 cm.

9. Let x be the increase in fare.

$R = 10 \cdot 300 \rightarrow R = (10+x)(300-15x) \rightarrow R = -15x^2 + 150x + 3000 \rightarrow$

$R = -15(x^2 - 10x + \_\_\_) + 3000 \rightarrow R = -15(x^2 - 10x + 25) + 3000 + 375 \rightarrow R = -15(x-5)^2 + 3375$

The most profitable fare to charge is $15.